Lecture Notes, Physics 105

Connor Duncan

May 16, 2019

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1 Math Review

1.1 Orthogonal Transformation

Changes basis of vectors to orthogonal basis. You know how to do this from linear algebra my guy.

In an orthogonal basis, recall that for every basis vector, $\lambda\lambda^\dagger = 1$

Imagine some vector $\vec{z} = \vec{z}\prime$ in some other coordinate system. Our change of coordinate matrix should be

$$\begin{bmatrix} x\prime \\ y\prime \\ z\prime \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

For a rotation about the z-axis.

Orthogonal transformation given by $\lambda\lambda^\dagger = 1$, where λ is the transformation matrix. what if we take $S \to S' \to S''$. Then $x' = \lambda x$, and $x'' = \lambda x' = w(\lambda x)$, where w is the change of coordinate matrix from $S' \to S''$.

Orthogonal operators form a *group*, i.e. multiplication of one orthogonal operator by another will produce another orthogonal operator, $\lambda, w \in G \to \lambda \cdot w \in G$.

Do they commute? No. $AB \neq BA$ in all cases.

Also know that $\det(\lambda\lambda^\dagger) = 1 = \det(\lambda)\det(\lambda^\dagger) = |\det(\lambda)|^2 \to \det\lambda = \pm 1$

1.2 Scalar, Vector Fields

Scalar Fields don't depend on coordinate system (invariant with respect to transformation), i.e. a number associated with every point in space.

Vector fields *do* depend on coordinate system. When you have $v_i' = \lambda_j v_j$, it satisifes coordinate transformations. (i.e. $\exists$ transformation matrix λ)

1.3 More orthogonal transformations

1.3.1 Group!

Orthogonal transformation Λ from coordinate system $S \to S'$, form a group, so that $\forall \Lambda, W$,

$$\Lambda W \neq W\Lambda$$
$$\Lambda\Lambda^\dagger = I_n$$
$$\Lambda_{ij} W_{jk} \neq W_{ij}\Lambda jk$$

the reason that we care it's a group is because it's closed under multiplication, i.e. for any orthogonal transformation Λ, W, their product $W\Lambda$ is also an orthogonal transformation.

1.3.2 $\det\Lambda = 1 \Leftrightarrow \Lambda$ has eigenvalues=1

$$(\Lambda - I)\Lambda^\dagger = 1 - \Lambda^\dagger = (1-\Lambda)^\dagger$$

Now, solve $||\Lambda_{ij} - a\delta_i j|| = 0$

$$||\Lambda - 1|| \cdot ||\Lambda^\dagger|| = ||(1-\Lambda)^\dagger|| = ||1-\Lambda||$$
$$\therefore$$
$$||\Lambda - 1|| = ||1 - \Lambda|| \to ||\Lambda - 1|| = 0$$

Where here, $1, I$ are used interchangeably to represent identity

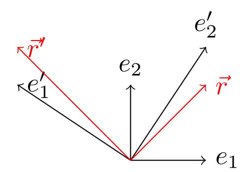
Consider operator $P|P_{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, it's the inversion operator, takes any vector $\vec{r} = (x, y, z) \to (-x, -y, -z)$. Determinant is -1, which allows P to be unitary (i.e. P^2=1)

We can now write any transformation Λ as a combination of rotation and inversion. Take $W = P\Lambda$, where W is a rotation matrix, since $||W|| = ||\Lambda||||P|| = 1$, then $PW = PP\Lambda = \Lambda$.

1.3.3 Eigenvectors

Consider the transformation

$$\begin{bmatrix} -1.5 & 1 \\ 1 & -1.5 \end{bmatrix}$$



We can consider either transformations of coordinate systems (i.e. basis vectors e_1, e_2) or of individual vectors ($\vec{r}$).

1.4 Rigid Body Motion

1.4.1 Vector Product (Cross)

Take vectors a_1, a_2 in the coordinate system defined with basis vectors e_1, e_2, e_3, so that $a_1 = (a_{11}, a_{12}, a_{13})$ and a_2 defined similarly.

$$|\vec{a_1} \times \vec{a_2}| = |\vec{a_1}| \cdot |\vec{a_2}| \sin(\theta)$$

Where θ is the angle between the two vectors.

When $S = \text{span}(\{e_1, e_2, e_3\})$, and is orthogonal basis.

$$e_1 \times e_2 = e_3 \qquad e_2 \times e_3 = e_1 \qquad e_3 \times e_1 = e_2$$

Levi-Civita tensor density defined by $[e_i \times e_j] = \epsilon_{ijk} e_k$, we can write the cyclic permutations of 1,2 and 3 to get the above identities regarding S.

We can also find the area of a parallelogram formed by two vectors a_1, a_2, it will be the square of the magnitude of the cross of these two vectors: $A = |a_1 \times a_2|^2$. We can also calculate this using

$$(a_1 \times a_2) = \left|\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right| = \det(a) e_3$$

1.4.2 Scalar Triple Product?

This man lectures very rapidly with lots of subscripts. Where's the professor? I'm pretty sure he's talking about the scalar triple product rn.

Want to prove that $\epsilon_{\alpha\beta\gamma} = \epsilon_{ijk}\Lambda_{\alpha i}\Lambda_{\beta j}\Lambda_{\gamma k}$. Alternatively we can show that $||\det A||\epsilon_{\alpha\beta\gamma} = \epsilon_{ijk}A_{\alpha i}A_{\beta j}A_{\gamma k}$.

$$a_3 \cdot (a_1 \times a_2) = V = ||a_{ij}|| = \left\|\begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & 0 \end{vmatrix}\right\|$$

double check that those zeroes are there. His handwriting was kind of scratchy here. Here's probably a better example `https://en.wikipedia.org/wiki/Triple_product`

Also, $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$, to be proven at home.

2 Newtonian Physics

2.1 Angular Velocity

Reintroduce angular velocity. Consider $\vec{r}$, with $\Lambda_{ij} = \delta_{ij} + \delta\varphi_{ij}$, with $\delta\varphi << 1$. We still want Λ to be unitary ($\Lambda\Lambda^\dagger = 1$), so

$$(1+\delta\varphi)(1+\delta\varphi^\dagger) = 11 + \delta\varphi + \delta\varphi^\dagger + \delta\varphi\delta\varphi^\dagger = 1$$

We know that $\delta\varphi = -\delta\varphi^\dagger$, since $\delta\varphi\delta\varphi^\dagger$ is very very small.

Now consider $x' = (1+\delta\varphi)x$. We can take

$$x' - x = \delta\varphi x$$

$$\delta x = \begin{bmatrix} 0 & -\delta\varphi_3 & \delta\varphi_2 \\ 0 & 0 & -\delta\varphi_1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\delta x = (\delta\varphi \times x)$$

In other words, $\delta r = [\delta\varphi \times r]$.

Then, $\frac{\partial x}{\partial t} = (\frac{\partial \phi}{\partial t} \times x \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = [\Omega \times r]$ where Ω is the *Angular Velocity*.

2.2 Linear Velocity

It's the time derivative of position. In arbitrary coordinates it's expressed simply

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t}$$

In cartesian coordinates it simplifies to the sum of the componentwise time derivatives.

2.3 Coordinate Transform w/ AV

Relations of coordinate transformation with some from S that has a very complicated motion compared to frame S'.

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)_S = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)_{S'} + (\Omega \times r)$$

3 Calculus of Variations

3.1 Minimzation

Recall from math 1B (or maybe 53), we're looking for this thing called geodesics, which takes the integral $\int_{x_1,y_1}^{x_2,y_2} dS$, and minimizing path length.

3.1.1 Ex: Bracchistocrone

Two points, A, B, and a particle falling in gravity, $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 0, m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = F_g$. The transit time from point $A \to B = t = \int_{x_1,y_2}^{x_2,y_2} \frac{dS}{v} = \int_{x_1,y_2}^{x_2,y_2} \frac{\sqrt{dx^2+dy^2}}{2yg}$. Integrating for y here is really really hard.

There's an easier way! Let $H = y'\frac{df}{dy'} - f$. We can find that $\frac{\mathrm{d}H}{\mathrm{d}x} = y''\frac{\mathrm{d}f}{\mathrm{d}y'} + y'\frac{\mathrm{d}}{\mathrm{d}x} - y'\frac{\mathrm{d}f}{\mathrm{d}y} - y''\frac{\mathrm{d}f}{\mathrm{d}y'} - \frac{\mathrm{d}f}{\mathrm{d}x}$, which all cancels to find that $\frac{\mathrm{d}H}{\mathrm{d}x} = -\frac{\mathrm{d}f}{\mathrm{d}x}$.

Basically, you integrate, do some fancy trig substitution and find that the solution is a cycloid. Bale's words here were explicitly "I'll let you do the dirty work here", so I guess this is to be proven at home

3.1.2 Ex: Plateau's Problem

Minimizing surface areas, w.r.t surface tension. Imagine two fixed rings, we want to minimize tension, so they'll pull each other together. Want to write a functional that corresponds to the energy of the system.

3.2 Euler-Lagrange Equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}f}{\mathrm{d}x'} - \frac{\mathrm{d}f}{\mathrm{d}x} = 0$$

describes the optimal path along some constraint, using a functional so that

$$dS = \int f dx$$

Recall that if f is not a function of the independent variable, (t in the expression above), then you can take

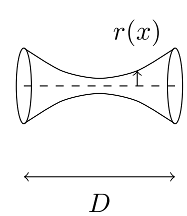
$$H = y'\frac{\mathrm{d}f}{\mathrm{d}y'} - f$$

and discover that

$$\frac{\mathrm{d}H}{\mathrm{d}x} = -\frac{\mathrm{d}f}{\mathrm{d}x}$$

3.2.1 Plateau's Problem, cont.

Take some soap film suspended between two hoops



surface area of a small band given by

$$dS = 2\pi r(x)\sqrt{1+\left(\frac{\mathrm{d}r}{\mathrm{d}x}\right)^2} dx$$

which implies that the total area is equal to

$$= 2\pi \int_{-D/2}^{D/2} r(x)\sqrt{1+r'(x)^2} dx$$

we can use the hamiltonian H to say that

$$H = r'\frac{\mathrm{d}f}{\mathrm{d}r} - f$$

with

$$\frac{\mathrm{d}f}{\mathrm{d}r'} = \frac{rr'}{\sqrt{1+r'^2}} \Rightarrow H = \frac{rr'^2}{\sqrt{1+r'^2}} - r\sqrt{1+r'^2}$$

Then, take $\frac{\mathrm{d}H}{\mathrm{d}x} = -\frac{\mathrm{d}f}{\mathrm{d}x}$, so $\frac{\mathrm{d}r}{\mathrm{d}x} = \pm\sqrt{\left(\frac{r}{H}\right)^2 - 1}$, os we integrate

$$\int \frac{dr}{\sqrt{\left(\frac{r}{h}\right)^2 - 1}}$$

we need to use hyperbolic cosines and sines, so we take $\frac{r}{H} = \cosh\psi$, which integrating gives

$$\int \frac{H \sinh\psi dr}{\sinh\psi} = H\psi$$

and thus, taking $\chi = \frac{H}{D}$[1].

$$r(\chi) = H\cosh\frac{\chi}{H}$$

where $\chi = \frac{D}{2}, r = R$. Final result comes out to be that

$$\frac{R}{D} = \chi\cosh\left(\frac{1}{2\chi}\right)$$

This is a number, which has to be equal to the geometry of the system!

3.3 Quantum $\Rightarrow$ Lagrangian Mechanics

Recall we have the transition amplitude, i.e. how probable it is to go from one state to another.

$$q_1(t_1) \to q_2(t_2)$$

would be expressed as

$$\langle q_1(t_1)|q_2(t_2)\rangle$$

which goes to

$$\langle q_2(t+\Delta t)|q_1(t)\rangle = \langle q_2| e^{-|\Delta H|} |q_1\rangle$$

we take action as

$$S(x(t)) = \int_{t_1}^{t_2} dt\left(\frac{p^2}{2m} - V\right)$$

then, with amplitude functional $A[x(t)] = e^{i\frac{S}{\hbar}}$, we can write an integral across every possible path, with

$$|A|^2 = \int_{\text{all paths}} x(t)e^{i\frac{S(x(t))}{\hbar}} dt$$

The path that wins is the one that oscillates the least, i.e. the one that has *stationary phase*, since integrating an oscillator gives zero. This means we want to find a *stationary* form of S, which is called *Hamilton's Principle*, which gives that

$$\delta S(x(t)) = S\int dt\left(\frac{p^2}{2m} - V\right) = 9$$

[1] don't worry, I don't totally understand how he did this integral either

4 Lagrangian Mechanics

4.1 Defining the Lagrangian

$$L = T - V$$

kinetic minus potential energy. Then, the lagrangian can be put into the Euler-lagrange equation to give that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}q'} - \frac{\mathrm{d}L}{\mathrm{d}q} = 0$$

which contains all of classical mechanics, since we can then write

$$S\int_{t_1}^{t_2} dtL(x, x', t)$$

with mass m, potential V, we have $T = \frac{mv^2}{2}$, so the lagrangian is

$$\frac{mv^2}{2} - V(q)$$

now, we write lagrange euler equation as

$$\frac{\mathrm{d}L}{\mathrm{d}q'} = mq' \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}q'} = mq'' \qquad \frac{\mathrm{d}L}{\mathrm{d}q} = \frac{\mathrm{d}V}{\mathrm{d}q}$$

which gives an equation of motion

$$mq'' = -\frac{\mathrm{d}V}{\mathrm{d}q} \Leftrightarrow F = ma$$

4.2 Ex: spherical pendulum

$$\begin{aligned}
\vec{r} &= l\cos\varphi\sin\theta\hat{x} + l\sin\varphi\sin\theta\hat{y} + l\cos\theta\hat{z} \\
\dot{\vec{r}} &= (-l\dot{\varphi}\sin\varphi\sin\theta + l\cos\varphi\dot{\theta}\cos\theta)\hat{x} \\
&\quad + (l\dot{\varphi}\cos\varphi\sin\theta + l\sin\varphi\dot{\theta}\cos\theta)\hat{y} \\
&\quad - l\dot{\theta}\sin\theta\hat{z} \\
\dot{\vec{r}}\dot{\vec{r}} &= l^2\dot{\varphi}^2\sin^2\theta + l^2\dot{\theta}^2
\end{aligned}$$

Then, we have

$$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}\left(ml^2\dot{\theta}^2 + ml^2\dot{\varphi}^2\sin^2\theta\right)$$

Which gives a lagrangian

$$L = T - V = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}\left(ml^2\dot{\theta}^2 + ml^2\dot{\varphi}^2\sin^2\theta\right) - mgl\cos\theta$$

Now, we want to apply the ELE, which gives two constraints

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}\dot{\theta}} - \frac{\mathrm{d}L}{\mathrm{d}\theta} &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}\dot{\varphi}} - \frac{\mathrm{d}L}{\mathrm{d}\varphi} &= 0
\end{aligned}$$

There's no φ dependence, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}L}{\mathrm{d}\dot{\varphi}}\right) = 0$$

which makes it a constant of motion, so we say

$$\left(\frac{\mathrm{d}L}{\mathrm{d}\dot{\varphi}}\right) = p_\varphi = ml^2\dot{\varphi}\sin^2\theta$$

We also have

$$\frac{\mathrm{d}L}{\mathrm{d}\theta} = mgl\sin\theta + ml^2\dot{\varphi}^2\sin\theta\cos\theta$$
$$\frac{\mathrm{d}L}{\mathrm{d}\dot{\theta}} = ml^2\dot{\theta} \to \frac{\mathrm{d}}{\mathrm{d}t}\{dvL\dot{\theta} = ml^2\ddot{\theta}$$

Then, we can find an equaiton of motion for θ, which gives

$$\ddot{\theta} = \frac{g}{l}\sin\theta + \varphi^2\sin\theta\cos\theta$$

with another term for $\dot{\varphi}$,

$$\dot{\varphi}^2 = \frac{p_\varphi^2}{m^2l^2\sin^4\theta}$$
$$\ddot{\theta} = \frac{g}{l}\sin\theta + \frac{p_\varphi^2}{m^2l^2}\frac{\cos\theta}{\sin^2\theta}$$

Integrating this is just rude. So let's analyze some cases.

4.2.1 case $\dot{\varphi} = 0$

implies $p_\varphi = 0$, which is then

$$\ddot{\theta} = \frac{g}{l}\sin\theta$$

which is just a regular harmonic oscillator

4.2.2 case $\dot{\theta}$ =constant

Implies that $\ddot{\theta} = 0$, which gives then that

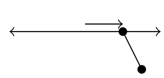
$$\frac{g}{l}\sin\theta + \dot{\varphi}^2\sin\theta_0\cos\theta_0 = 0$$
$$\left(\frac{g}{l} + \dot{\varphi}^2\cos\theta_0\right)\sin\theta_0 = 0$$

if $\theta_0 = 0, \pi$ etc, then the pendulum is just balanced at the top, not moving.

if $\cos\theta_0 < 0$, we have $\theta_0 > \pi/2$, which gives that $\dot{\varphi}^2 > \frac{g}{l} = \omega_0$

We could also integrate this and get complex motion, but these are the stable forms.

4.3 Driven Pendulum



Point on axis is being pushed, what is the motion of the point at the bottom of the pendulum?

$$x = a\cos\omega t$$
$$r = x\hat{x} + l(\sin\theta\hat{x} - \cos\theta\hat{y})$$
$$\dot{\vec{r}} = (\dot{x} + l\dot{\theta}\cos\theta)\hat{x} + l\dot{\theta}\sin\theta\hat{y}$$
$$\dot{r}^2 = \dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}l\cos\theta$$

we set up the lagrangian

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}^2\dot{\theta}^2 l\cos\theta) + mgl\cos\theta$$

and try to find simple solutions.

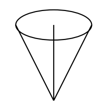
4.4 Examples of Lagrangian Mechanics

4.4.1 Cone?

Missed the first one, but we know that *angular momentum is conserved.* Basically, just a whole lot of algebran happening here, with something rotating in a conic shape, or on the surface of a cone (maybe like throwing a coin into one of those things at McDonalds).

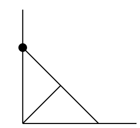
Stable solutions can be given by $\ddot{r} = 0$. I.e. the coin doesn't ever go into the money receptacle.

Gives that $\dot{\theta}^2\tan\alpha = \frac{g}{r_0} \Rightarrow \dot{\theta}^2 = \frac{\omega_0}{\tan\alpha}$ Which implies that in order to have some stable orbit in a cone at a certain angle α, you have explicit angular momentum dependence.



4.4.2 Mass/Spring on a T

Imagine some T on a tabletop, that looks a bit like this



Where the dot is connected to the T by a spring that's hooked up at the juncture.

Let ωt be the angle between the x-axis and the T.

then we can write $\vec{r} = (l\cos\omega t - \rho\sin\omega t)\hat{x} + (l\sin\omega t + \rho\cos\omega t)\hat{y}$

$$T = \frac{1}{2}m(\dot{r}\dot{r}) = \frac{1}{2}m(\omega^2(l^2 + \rho^2) + \dot{\rho}^2 + 2\omega l\dot{\rho})$$

put in to the euler lagrange equation

$$\frac{\partial L}{\partial \rho} = m\omega^2\rho - k\rho$$
$$\frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho} + \omega l$$

So, equating these two things, gives us that

$$\ddot{\rho} + (\frac{k}{m} - \omega^2)\rho = 0$$

which yields 3 sort of 'classes' of solutions. First is where $\omega < \sqrt{\frac{k}{m}}$, which yields a simple harmonic oscillator, very fun!

We also could have $\omega > \sqrt{\frac{k}{m}}$, which gives us that $\rho(t) = Be^{\alpha t} + Ce^{-\alpha t}$.

There's also the case of equality, which gives us resonant oscillation, or just a growth term $\rho(t) \sim t$.

4.4.3 Now with Gravity!

Take the previous problem, and just add gravity into the mix, since we all like to have fun.

Now we have $V = mgy$, and we have y from the previous problem, so the lagrangian becomes some really long wild thing, that I cannot see (Bale didn't do the whole thing out, but the principle of the problem is similar to what we did above).

5 Symmetries (Formally)

Considering changes to L (the lagrangian) when we perturb one of the coordinates. Say it's $q_i \to \tilde{q}_i = q_i + k_i$

5.1 Linear Momentum

consider

$$\tilde{x} = x + \epsilon \qquad \dot{\tilde{x}} = \dot{x}$$

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\dot{\tilde{x}}^2$$

$$L = \frac{1}{2}m\dot{x}^2 - V(x) \qquad \tilde{L} = \frac{1}{2}m\dot{\tilde{x}}^2 - \tilde{V}(x)$$

If we apply the constraint that $L(x, \dot{x}) = \tilde{L}(\tilde{x}, \dot{\tilde{x}})$, then $V(x)$ has to be invariant to spatial pertubation, which implies that $F_x = 0$, since $-F = \frac{\partial V}{\partial x}$.

This is just a meme'd way of writing conservation of linear momentum, since it boils down to

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = 0$$

5.2 Noethers Theorem (intro)

$$L(q, \dot{q}) = L(q + \epsilon k, \dot{q} + \epsilon \dot{k})$$

$$L(q, \dot{q}) = L(q + \epsilon k, \dot{q} + \epsilon \dot{k}) = L(q, \dot{q}) + \epsilon \sum_i \dot{k}_i \frac{\partial L}{\partial \dot{q}_i} + \epsilon \sum_i k \frac{\partial L}{\partial q_i} ....$$

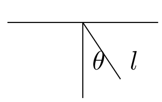
This just applies the constraint that the sum of the first n taylor expanded terms has to be zero, which is *Noethers Theorem.*

Another, simpler way of writing this is that

$$\sum K \frac{\partial L}{\partial \dot{q}} = C$$

where C is constant.

Example



we just apply

$$L = \frac{1}{2}m\dot{l}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$$

And derive properties from this, like conservation of angular momentum.

5.3 Probably the Hamiltonian

consider

$$H = \dot{q}\frac{\partial L}{\partial \dot{q}} - L$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{dL}{dt}$$

some long thing using the chain rule. has it simplify down to the above form, which implies htat H is a conserved quantity with respect to time.

Take

$$L = \frac{1}{2}m\dot{x}^2 - V(x)$$

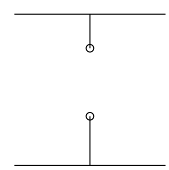
$$H = \dot{x}\frac{dL}{d\dot{x}} - L = m\dot{x}^2 - (\frac{1}{2}m\dot{x}^2 + V(x)) = \text{total energy}$$

which lets us say H is a total energy, the *hamiltonian.*

6 More Lagrangian Mechanics

6.1 Pertubations w/ a pendulum

Imagine two pendulums as follows



small pertubations will be stable for the top pendulum in gravity, about $\theta = 0$, but unstable for the lower.

General solution for for θ at 0 is given as

$$\delta\theta = A_1 e^{-i\omega_0 t} + A_2 e^{i\omega_0 t}$$

if $\theta = \pi$, we find

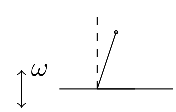
$$\delta\ddot{\theta} = \omega_0^2 \delta\theta$$

which gives

$$\delta\theta = A_1 e^{-\omega_0 t} + A_2 e^{-\omega_0 t}$$

This comes from finding solutions to differential equations. You probably should have taken 54 as a prerequisite to this class, but if u didn't hmu for textbooks (totally legal i promise[2]).

6.2 Interesting thing to do at home



ground is oscillating, with $\omega >> \omega_0$ of the pendulum,

6.3 More general

take some lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 - V(q)$$

then equilibrium given by

$$\frac{\partial V}{\partial q} = 0 \to q = q_0$$

[2] no promises

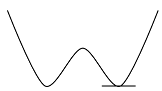
Take some $q = q_0 + \delta q$, then the lagrangian is given

$$\mathcal{L} = \frac{1}{2}(\delta \dot{q})^2 - V(q_0 + \delta q) = \frac{1}{2}\delta\dot{q}^2 - \frac{1}{2}\left(\frac{\partial^2 V}{\partial q^2}\right)_{q=q_0} \partial q^2$$

$$\partial \ddot{q} = -\left(\frac{\partial^2 V}{\partial q^2}\right)_0 \delta q$$

$$\delta q = Ae^{i\omega t} \qquad \omega^2 = \frac{\partial^2 V}{\partial q^2}_0 \geq 0$$

which gives the sort of intuitive solution that there needs to be a potential well around stable systems, i.e.



where the curve represents potential.

6.4 Two Pendulum system

I'm not going to do a drawing of this one. Length of p1 is l_1, l_2, angle to vertical given by θ_n, where n represents the penulum number, similarly for the mass. We have

$$U = -m_1 g l_2 \cos\theta_1 - m_2 g(l_2 \cos\theta_1 + l_2 \cos\theta_2)$$
$$T = \frac{1}{2}m_1(\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2}m_2(\dot{x_2}^2 + \dot{y_2}^2)$$

example for x_2, y_2: given by

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$
$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$

We get some really really really long for for the lagrangian, which it's almost certainly too long to type. Just do the derivative, and you get that it's

$$\mathcal{L} = T - U$$

The solution is then given for

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{d\mathcal{L}}{\theta_i}\right) = \frac{\partial \mathcal{L}}{\partial \theta_i}$$

only equilibrium point is given for $\theta_i = 0 \forall i$. Even if the pendulums were perpendicular to each other, it would be in unstable equilibrium.

Now, we want to linearize the system so that there are only quadratic terms in the lagrangian. No higher powers than 2.

for small θ, we just taylor expand everything, gives us

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)l_2^2\dot{\theta_1}^2 + \frac{1}{2}m_2 l_2^2 \dot{\theta_2}^2$$
$$+ m_2 l_1 l_2 \dot{\theta_1}\dot{\theta_2} + \frac{1}{2}(m_1 + m_2) g l_1 \theta_1^2 + \frac{1}{2} m_2 g l_2 \theta_2^2$$

We can simplify this down, wiriting

$$\mathcal{L} = \frac{1}{2}\dot{\theta_1}^2 + \frac{1}{2}\mu l^2 \dot{\theta_2}^2 + \mu l \dot{\theta_1}\dot{\theta_2} - \frac{1}{2}\omega_0^2 \theta_1 \theta_1^2 - \frac{1}{2}\mu l \omega_0^2 \theta_2^2$$

with

$$\mu = \frac{m_2}{m_1 + m_2} \qquad l = \frac{l_2}{l_1}$$

This yields

$$\ddot{\theta_1} + \mu l \ddot{\theta_2} = -\omega_2^2 \theta_1$$
$$l\ddot{\theta_2} + \ddot{\theta_1} = -\omega_0^2 \theta_2$$

Now, we rewrite in matrix form, trying to find $\theta_i = A_i e^{i\omega t}$.

$$\begin{bmatrix} \omega_0^2 - \omega^2 & -\mu l \omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 l \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

Let's call that matrix D, and take it's determinant, to see if either A_1, A_2 must be zero.

$$||D|| = (1-\mu) l \omega^4 - \omega^2 \omega_0^2 (1 + l) + \omega_0^4$$

The solutions then, are given as

$$\omega_\pm^2 = \frac{\omega_0^2(1 - l) \pm \sqrt{(1 + l)^2 - 4(1 - \mu) l}}{2(1-\mu) l}$$

Now, we consider A_-, A_+, then, we can write

$$\theta_y = C_y^+ A_y^+ e^{i\omega_+ t} + C_y^- A_y^- e^{i\omega_- t}$$

with C_i as the initial condition.

1. Equilibrium
2. Linearize
3. $\omega \to A_i$
4. something else

These represent the normal modes of a system.

Lets consider the equation with the matrix D for the case of ω_+. It's provable that $\omega_+^2 > \omega_0^2$, so we take

$$(\omega_0^2 - \omega_+^2) A_1 = \mu l \omega_+^2 A_2$$

$$A_1 = \frac{\mu l \omega_+^2}{\omega_0^2 - \omega_+^2} A_2 \qquad \text{sign}\,\frac{A_1}{A_2} = -1$$

6.5 Four points on circle

Take four points on a circle, all of which are connected by springs of coefficient k. (I might add a drawing of this later, it's kind of hard to picture). All points are of the same mass.

$$\mathcal{L} = \frac{1}{2} m R^2 \sum_{i=1}^{4} \dot{\varphi_i}^2 -$$
$$\frac{1}{2} k \times R^2 \left[(\varphi_1 - \varphi_2)^2 + (\varphi_1 - \varphi_4)^2 + (\varphi_2 - \varphi_3)^2 + (\varphi_2 - \varphi_4)^2\right]$$

Which gives a bunch of coupled oscillatros, for $\ddot{\varphi}_i$. It's more convenient to write them as a giant matrix

$$\begin{bmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 & 0 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 & -\omega_0^2 & 0 \\ 0 & -\omega_0^2 & -2\omega_0^2 - \omega^2 - \omega_0^2 & \\ -\omega_0^2 & 0 & -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0$$

We compute the determinant, and find that

$$||D|| = \pm(2\omega_0^2 - \omega^2)(4\omega_0^2 - \omega^2)\omega^2 = 0$$

There are some eigenmodes,

$$\omega = 0 \qquad \omega = 2\omega_0 \qquad \omega = \sqrt{2}\omega_0$$

7 Forced, Damped Oscillators

Try some $F(t) = C_0 e^{-i\omega t}$, then we can think about something with a harmonic sollution, i.e. $Z(t) = \tilde{A}e^{-i\omega t}$, with motion described as $\mathrm{Re}\{z(t)\}$. We punch this into the general solution (homogeneous+inhomogenous), to get an answer.[3]

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2\right) z(t) = F(t)/m$$

Then, we solve for $\tilde{A} = \frac{C_0}{-\omega_0^2 - 2i\beta + \omega^2}$. Particular term is complex!

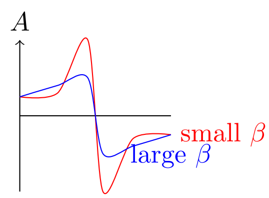
$$\tilde{A} = \frac{\frac{F_0}{m}}{\omega_0^2 - 2i\beta\omega - \omega^2} = \frac{\frac{F_0}{m}}{\omega_0^2 - \omega^2 - 2\beta i\omega}$$

You also get elasitic and absorptive amplitude.

$$A = \frac{F_0/m(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \equiv \text{elastic}$$
$$B = \frac{2\beta\omega F_0/m}{(\omega_0^2 - \omega^2 + 4\beta^2\omega^2} \equiv \text{absorptive}$$

which gives

$$z(t) = A\cos\omega t + B\sin\omega t + i(B\cos\omega t - A\sin\omega t)$$



constant power$\sim \frac{\text{work}}{\text{time}}$.

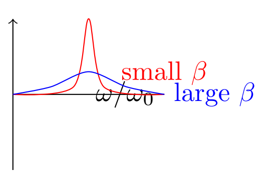
$$P(t) = F(t)\cdot\dot{z}(t)$$
$$= F_0\cos\omega t \cdot \frac{\partial}{\partial t}$$
$$= F_0\cos\omega t(-\omega A\sin\omega t + \omega B\cos\omega t)$$

The first term averages to zero in half a cycle, which is why we call it the elastic amplitude.

The average of B over a half cycle represents the energy dissapated by the system, which is the same as $\beta = \frac{1}{2}F_0\omega B$.

$$\mathrm{Re}(Z) = \frac{A}{\sqrt{(\omega_0^2 - \omega^2) + 4\beta^2\omega^2}} cos(\omega t - \varphi)$$

With $\varphi = \tan^- 1\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$.



Define $Q \equiv$quality factor.

$$Q = \frac{\omega}{2\beta} = \frac{\sqrt{\omega_0^2 - \beta^2}}{2\beta}$$
$$\ddot{z} + \frac{\dot{z}}{Q} + z = 0$$

Low Q correspond to a lot of damping (i.e. broad resonance), and a high Q corresponds to little damping.

Check out Oscillations and Waves by A.P. French.

There are also these green functions, which are kind of fun. It's like an optics or quantum problem, when you're crossing a barrier.

$$\ddot{z} - 2\beta\dot{z} + \omega_0^2 z = F_0/m \qquad \text{Region I}$$
$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = 0 \qquad \text{Region II}$$

You go about solving this using the typical method of solving differential equations, which gives you (in region I)

$$C_2 = \frac{\beta}{m}C_1 = \frac{\beta}{\omega_1}\frac{F_0}{m\omega_0^2}$$

Which gives

$$z(t) = \frac{F_0}{m\omega_0^2}\left[1 - e^{-\beta t}\cos\omega_1 t - \frac{\beta}{m}e^{-\beta t}\sin\omega_1 t\right]$$

between $0 \leq t \leq \tau$ (boundary). Taylor expanding, we can write this as

$$z(t) = \frac{F_0}{m\omega_0^2}\left[1 - (1 - \beta t + \frac{(\beta t)^2}{2}...\right]$$

which simplifies down to

$$\approx \frac{F_0}{m\omega_0^2}\left(\frac{\omega_0^2 t^2}{2} - \frac{\beta t^2}{2}\right)$$

which tells $z \sim t^2$ for early times, which is about what we'd expect.

We also have, at the boundary (a homogeneous solution starting at $t = \tau$).

$$z_0(t) = e^{-\beta(t-\tau)}\left[D_1\cos\omega_1(t-\tau) + D_2\sin\omega_1(t-\tau)\right]$$

Think of it like integrating over a bajillion little impulses. No testing on green functions apparently (it isn't in the book).

$$z(t) = \int_{t'\to-\infty}^{t} G(t,t')F(t')dt'$$

with $F(t')$ the forcing function, and G the green function.

Green function of damped SHO with $z(0) = \dot{z}(0) = 0$:

$$G(t,t') = \frac{e^{-\beta(t-t')}\sin\omega_1(t-t')}{m\omega_1}$$

DO AT HOME, try taking $F(t') = \alpha t'$ (linearly increasing force), try doing the integral with the forcing function.

8 Central Force Motion

Recall, newton, we have $F = m\vec{a} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$. So, for some direction $\hat{s}$, if we have $F\cdot\hat{s} = 0$, then $\vec{p}\cdot\vec{s} = 0$, also true for torque. Implies conservation of true, also true for angular momentum.

Definition 8.1. Central force is a force such that $\vec{F}(\vec{r}) = f(r)\hat{r}$, i.e. force depends only on vector between objects.

This means that the torque $\vec{r}\times\vec{F} = \vec{r}\hat{r}f(r) = 0$, which implies that $\vec{L} \equiv$ constant.

[3] Bale said 'read the fourier thing' which I assume is a reference to the text

8.1 Two Body Problem

let r_1, r_2 be vectors pointing to two masses, m_1, m_2 respectively. let $\vec{R}$ be the vector pointing to the center of mass, and $\vec{r}$ be the vector pointing from m_1 to m_2. Let r'_n be the vector pointing from m_n to the cm.

We can think about the lagrangian. noting that

$$\vec{r_n} = \vec{R} + \vec{r_n}'$$

Now, writing down the component bits of T, we have

$$T = \frac{1}{2}(m_1 \dot{\vec{r_1}}^2 + m_2 \dot{\vec{r_2}}^2)$$

which expands (after some substitution) to

$$T = \frac{1}{2}\left(m_1 \dot{R}^2 + m_1 \dot{r}_1'^2 + 2m_1 \dot{R}\dot{r}_1' + m_2 \dot{R}^2 +_{,2} \dot{r}_2'^2 + 2m_2 \dot{R}\dot{r}_2'\right)$$

If we define the center of mass as

$$\sum_i m_i r_i = \sum_i m_i \vec{R}$$

then the cross terms from our dot product cancel, and T simplifies to

$$T = \frac{1}{2}(m_1 + m_2)\dot{R}^2 + \frac{1}{2} m_1 \dot{r'}_1^2 + \frac{1}{2} m_2 \dot{r'}_2^2$$

we also have

$$\vec{r'}_2 = \frac{-m_1}{m_1 + m_2}\vec{r} \qquad \vec{r'}_1 = \frac{m_2}{m_1 + m_2}\vec{r}$$

The lagrangian then becomes, using this simplification for the reduced mass

$$\frac{1}{2}(m_1 \dot{r'}_1^2 + m_2 \dot{r'}_2^2) = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\dot{r}^2$$

We can call this reduced mass μ, and the total mass M. Now, we get

$$T = \frac{1}{2}(M\dot{R}^2 + \mu\dot{r}^2)$$

Now, the lagrangian

$$\mathcal{L} = T - U = \frac{1}{2}(M\dot{R}^2 + \mu\dot{r}^2) - U(r)$$

Immediately, we can tell that R is cyclic, (i.e. $\frac{\partial L}{\partial R} = 0$, which implies that $m\dot{R} \equiv$constant, which can be edrived from the euler-lagrange equations easily)

Since the momentum of the center of mass is conserved, we're just going to drop the $M\dot{R}^2$ term, since we're just changing to a frame that's moving with the center of mass.

The problem is now basically a single-body problem.

Conservative forces that depend on only r, so we have $F(r) = f(r)\hat{r}$[4] so

$$\vec{F} = -\vec{\nabla}V(r) = f(r)\hat{r}$$

$$V(r) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}')d\vec{r}'$$

We can convert this to a 2-d problem in polar coordinates, with the knowledge that $\frac{\mathrm{d}L}{\mathrm{d}t} = 0$, so, we can write $\mathcal{L}$ as

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

[4]note that I'm dropping a lot of over-arrows, but these objects are still vectors

We need to choose a form for the potential, take

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = 0 \qquad l = mr^2\dot{\theta} \equiv \text{angular momentum}$$

This immediately yields one of keplers laws, since

$$\frac{\mathrm{d}}{\mathrm{d}t} = 0$$

is the areal velocity, we get keplers second law, since that tells you the area swept out by a radius vector per unit time is always the same.

This gives the equation of motion

$$m\ddot{r} - mr\dot{\theta}^2 = f(r)$$

with the knowledge that $\dot{\theta} = \frac{l}{mr^2}$, we can write now that

$$m\ddot{r} - \frac{l^2}{mr^3} = f(r)$$

which is a one dimensional equaiton of motion, which we know how to solve. Total energy is

$$E = \frac{1}{2}(m\dot{r}^2 + \frac{l^2}{mr^3} + V(r) \equiv \text{const}$$

Let's integrate the equation of motion

$$\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{l}{mr^2}$$

$$\int d\theta = \int \frac{l}{mr(r)^2}dt$$

$$\Delta\theta = l\int_0^t \frac{dt}{mr^2(t)}$$

We also have

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}$$

which gives

$$t = \int dt = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}}$$

So, what can we qualitatively get out of this problem? WE STILL HAVENT SPECIFIED THE POTENTIAL :eyeroll:.

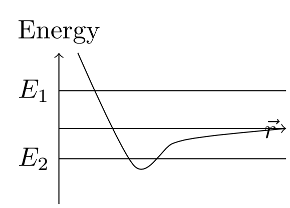
Let's define an *effective potential*. Let $V'(r) = V(r) + \frac{l^2}{+2mr^2}$, then we can rewrite energy as

$$E = \frac{1}{2}m\dot{r}^2 + V'(r)$$

Finally, we should take $V = -\frac{k}{r}$ (like gravity, or a coulomb force). Then,

$$V'(r) = \frac{-k}{r} + \frac{l^2}{2mr^2}$$

This looks qualitatively like the following



Between the certain energy levels, there is different behavior. E_1 corresponds ot a hyperbola, since ther's only one turning point, it will make one turn, which is a path that looks like

at E_2, it will be an ellipse, from r_1 to r_2 (i.e. the places E_2 intersectes energy).

At E_4, you should get a parabola (lowest point on the energy graph).

We can also solve for $\theta(t), r(t)$ using L, E are constant. In principle, we coould solve and make plots in terms of time etc, but we also want to see plots of $r(\theta, \theta(r)$. We can use conservation of angualr momentum to achieve this goal. $\frac{d\theta}{dt} = \frac{l}{mr^2}$, which gives us $ldt = mr^2 d\theta$.

We can write this as

$$d\theta = \frac{ldr}{mr^2\sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}}$$

with potential written as inverse r, we get

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{l}{r^2}}}$$

making a u-sub, for $r = \frac{1}{u}$, we get

$$\theta = \theta_0 \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - u^2}}$$

To be continued thursday.

8.2 Central Force Motion, Continued

Recall, we have $m\ddot{r} - \frac{l^2}{mr^3} = f(r)$, with constant energy

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} + V(r) \equiv \text{constant}$$

which allows us to write

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{l}{mr^2}$$
$$d\theta = \frac{ldr}{mr^2\sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}}$$
$$\theta = \theta_0 + \int_{r_0}^{r} \frac{dr}{mr^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}}}$$
$$\theta = \theta_0 + \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mV(u)}{l^2} - u^2}}$$

If we let $f \sim \frac{1}{r^2}$, we get that, with k as a coupling constant (i.e. how much force is scaled by).

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + \sqrt{1 + \frac{2El^2}{mk^2}}\cos(\theta - \theta_0)\right)$$

an example of k for gravity is $V = \frac{GmM}{r}$, leaves $k = GmM$.

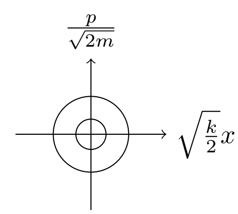
8.2.1 Kepler Orbits/Phase Diagrams

$$\frac{1}{r} = c\left(1 + \epsilon\cos(\theta - \theta_0)\right)$$
$$\epsilon = \sqrt{1 + \frac{2El^2}{mk^2}} \equiv \text{eccentricity of orbit}$$

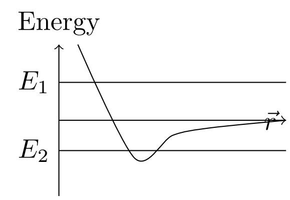
Now, we want to think about phase diagrams. We have

$$(\sqrt{E})^2 = \left(\frac{p}{\sqrt{2m}}\right)^2 + \left(\sqrt{\frac{k}{2}}x\right)^2$$

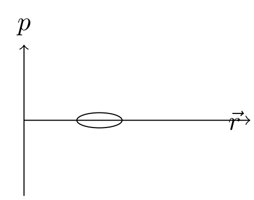
So particles move on circles in this abstract phase space



Recall our diagram of energy from the previous lecture,



It has a corresponding phase diagram,



This is a *suuuuper rough approximation* that you should verify on your own using python or desmos or something. Use $V \sim \frac{1}{r^2} - \frac{s}{r}$ where s is a constant.

If curves on the phase diagram are closed, they're *trapped solutions*, i.e. they want to stay within the potential well.

For given eccentricity (let $\cos\theta = 1$) we can calculate the minimum r of the orbit in a straightforward manner.

$$r_{min} = \frac{l^2}{mk(1+\epsilon)}$$

and maximum r, we have

$$r_{max} = \frac{l^2}{mk(1-\epsilon)}$$

There are also the unbound orbits, which give you

$$\frac{1}{r} = C(1 + \epsilon\cos(\theta - \theta_0))$$

the right hand side can be zero, which means $r_{max} \to \infty$.

Let's examine the orbits. First, let $r = \frac{1}{\alpha}(1 + \epsilon\cos\theta)$, which gives $\alpha = r + \epsilon r\cos\theta = r + \epsilon x$. Then, we get

- $\epsilon = 0 \to \frac{1}{r} = \frac{mk}{l^2}$ which gives constant r, and is thus a circle. Also note it would give $x^2 + y^2 = \alpha^2$ which also describes a circle.

- $0 < \epsilon < 1 \to 0 < 1 + \frac{2El^2}{mk^2} < 1 \to \frac{-mk^2}{2l^2} < E < 0$. This case corresponds to the area of our energy diagram beneath $E = 0$.

  Completing the square, we alsonote that $\frac{\left(x+\frac{\alpha\epsilon}{1-\epsilon^2}\right)^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a = \frac{\alpha}{1-\epsilon^2}, b = \frac{\alpha}{\sqrt{1-\epsilon^2}}$. a is called the *semimajor axis*, and b the *semiminor axis*. ϵ is a unitless quantity. This centers the ellipse at $x_0 = -\frac{\alpha\epsilon}{1-\epsilon^2}$. The ellipse also has a *focus*, with $c^2+b^2 = a^2$, c being the focus. You can solve it to be $c = \frac{\alpha\epsilon}{1-\epsilon^2}$, and another focus at the origin.

- $\epsilon = 1$. We get $y^2 = \alpha^2 - 2\alpha x$, since the x^2 terms from our other equation cancel, which gives the parabola $y^2 = -2\alpha\left(x - \frac{\alpha}{2}\right)$.

- $\epsilon > 1$. We find $\frac{\left(x-\frac{\alpha\epsilon}{\epsilon^2-1}\right)^2}{a^2} - \frac{y^2}{b^2} = 1$, which gives a hyperbola!

For a better explanation of what's happening here/pictures, see Taylor fig. 8.11.

8.2.2 Keplers Laws

This lets us derive keplers laws

1. Planets Move in ellipses with one focus at the sun (equiv to condition $0 \leq \epsilon < 1$)

2. Radius vector sweeps out equal area at equal time (equiv to conservation of momentum $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2m}$.

3. The square of the period of an orbit (T) is proportional to the cube of the semimajor axis ($T^2 = \frac{4\pi^2 a^3}{Gm_0}$). This can be shown with $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{l}{2m} \to A = \frac{l}{2m}T = \pi ab \equiv$area of ellipse.

Energy diagram, recall from previous week where the minimum with one r is a circular orbit, i.e. $\frac{\mathrm{d}V}{\mathrm{d}r_0} = 0$, question is only whether or not its stable, i.e. a relative minimum or maximum. there was already a homework problem on this, so I would watch out if I were you!

8.3 Ex: find stable circular orbits (this is very similar to the homework!)

$F(r) = \frac{-k}{r^n}$, which gives $V(r) = \frac{k}{(n-1)r^{n-1}}$, with effective potential $V_e(r) = \frac{l^2}{2mr^2} - \frac{k}{(n)r^{n-1}}$.

Stable point of this? $\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{k}{r^n} - \frac{l^2}{mr^3} = 0$ which gives $r_0 = \left(\frac{mk}{l^2}\right)^{-n+3}$. So, for stable orbits, $\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{-nk}{r^{n+1}} + \frac{3l^2}{mr^4} > 0$, which gives $(3-n)\frac{l^2}{m} > 0$ which implies that $n < 3$ must be the case for stable circular orbits.

If we want pertubations around a stable orbit, we can taylor expand aroudn the equilibrium point (r_0). We have the equation of motion

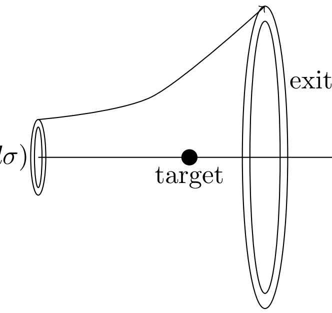
$$m\ddot{r} - \frac{l^2}{mr^2} = -f(x) = -\frac{\mathrm{d}L}{\mathrm{d}r}$$

At this point, bale kind of gave up because he doesn't want to give away the answer to the homework, but hit me up if u have any questions, the algebra SUCKS.

9 Hamiltonian Mechanics

9.1 Rutherford Scattering

We're talking about quantum mechanics rn, because the hamiltonian is very imortant. Rutheford scattering, observed really large alpha particle scattering which was weird.



Some differential cross section of the angle between the horizontal and scattering at infinity is given as

$$d\sigma(\text{into}d\Omega) = \frac{d\sigma}{d\Omega}d\Omega$$

by into $d\Omega$ I have no idea what he measn yet. It will become clear.

$$\sigma = \int \frac{d\sigma}{d\Omega}d\Omega = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi \frac{d\sigma}{d\Omega}(\theta, \varphi)$$

we could try doing this with conservation of angular momentum, that $l = mv_0 s$.

Or, take the total energy $E = \frac{1}{2}mv_0^2$ where v_0 in both represents the velocity at $r = -\infty$. Eliminating v_0, we have $l = \sqrt{2mE}$. Integrating this for the impluse, we get

$$\Psi = \int_{r_1}^{\infty} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} + \frac{2mV(r)}{l^2} - \frac{1}{r^2}}}$$

We can just cut to the chase, and say there's a hyperbolic orbit with central forcing, of which we already have an equation (doesn't this picture look kind of familiar?).

$$k = -z_1 z_2 e^2$$

deno ting coulomb force, then we can write (double check this in the book, I was way too far away to see this coherently)

$$\frac{1}{r} = -\frac{mz_1z_2e^2}{l^2}(1 + \epsilon\cos\psi)$$

So, as $r \to \infty$, we have $\frac{1}{r} \to 0 = \frac{mz_1z_2e^2}{l^2}(\epsilon\cos\psi + 1)$. This gives that

$$\cos^2\frac{\theta}{2} = \frac{\epsilon^2 - 1}{\epsilon^2}$$

which gives

$$\cot^2\frac{\theta}{2} = \epsilon^2 - 1 = \frac{2Es}{z_1z_2e^2}$$

Now, we have a solution for s,

$$s = \frac{z_1z_2e^2}{2E}\cot^2\frac{\theta}{2}$$

which gives

$$\sigma(\theta) = \frac{1}{4}\left(\frac{z_1z_2e^2}{2E}\right)^2 \csc^4\frac{\theta}{2} = \frac{s}{\sin\theta}\frac{\mathrm{d}s}{\mathrm{d}\theta}$$

This was a super messy derivation of this idea, so I would probably look it up/check back into a textbook for a better idea, especially because notation wasn't super consistent/things were p hard to see.

9.2 Beginning Hamiltonian Mechanics

Recall we defined $\mathcal{L}(q, \dot{q}, t) = T - V$, with the appropriate euler lagrange equations.

We also defined generalized momentum, $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$.

The hamiltonian formulation takes out the $\dot{q}$ dependence and replaces it with p.

$$H = \sum_i p_i \dot{q}_i - \mathcal{L}(q, \dot{q}(q, p), t) = H(q, p, t)$$

We're now working in a $2n$ dimensional "phase space" (p, q).

Dropping i', we have

$$\begin{aligned}\frac{\partial H}{\partial p} &= \frac{\partial}{\partial p}\left(\sum_i p_i \dot{q}_i - \mathcal{L}(q, \dot{q}(q, p), t)\right) \\ &= \dot{q} + p\frac{\partial \dot{q}}{\partial p} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial p} \\ &= \dot{q} \\ \frac{\partial H}{\partial q} &= \frac{\partial}{\partial q}\left(\sum_i p_i \dot{q}_i - \mathcal{L}(q, \dot{q}(q, p), t)\right) \\ &= p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q} \\ &= -\frac{\partial \mathcal{L}}{\partial q}\end{aligned}$$

which, by the euler-lagrange equations gives

$$\frac{\partial H}{\partial q} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = -\frac{\mathrm{d}}{\mathrm{d}t}q$$

Finally, we might also take

$$\begin{aligned}\frac{\partial H}{\partial t} &= \frac{\partial}{\partial t}\left(\sum_i p_i \dot{q}_i - \mathcal{L}(q, \dot{q}(q, p), t)\right) \\ &= p\frac{\partial \dot{q}}{\partial t} - \frac{\partial \mathcal{L}}{\partial \dot{q}}...\end{aligned}$$

he goes too fast lmao.

Gives **Hamiltons Equations of Motion**

$$\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial t} &= -\frac{\partial \mathcal{L}}{\partial t}\end{aligned}$$

9.3 ex: hamiltonian, sho

We can do some fancy algrebra to derive that for a simple harmonic oscillator, we get

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

Note from Connor, ur friendly DSP boi, this is super similar to quantum mechanics hamiltonian!

9.4 ex: Particle, Magnetic field

Trust and verify that the lagrangian is written as

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 - e\varphi(r, t) + \frac{e}{c}\dot{\vec{r}} \cdot \vec{A}(r, t)$$

with $B = \vec{\nabla} \times \vec{A}$, and $\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}$.

Basically just do the math out. I swear, I cannot see the board on the other side of the room, so I'll try and do this out on my own, or talk to somebody in the class to get this example.

Hamiltons equations of motion: also gives u ray tracing, from optics.

9.5 Recall

Recall that we can write the hamiltonian in terms of the lagrangian as $H = p\dot{q} - \mathcal{L}(q, \dot{q}(q, p), t)$, with $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$, called the canonical momentum.

Using the euler lagrange equation, we derived

$$\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial t} &= -\frac{\partial \mathcal{L}}{\partial t}\end{aligned}$$

which are hamiltons equations of motions, which are *all first order equations* which is convenient!.

9.6 Stationary Phase Derivation

Recall

$$S = \int_{t_1}^{t_2} \mathcal{L} dt$$

+ we used to minimize this bad boy. Now, let

$$S = \int_{t_1}^{t_2} (p\dot{q} - H)\, dt$$

Now, using calculus of variations, let's replace $q \to q + \epsilon\eta, p \to p + \epsilon\chi$, with $\eta(t_1) = \chi(t_1) = \eta(t_2) = \chi(t_2) = 0$, giving

$$S = \int_{t_1}^{t_2} \left((p + \epsilon\chi)(\dot{q} + \epsilon\dot{\eta}) - H(q + \epsilon\eta, p + \epsilon\chi, t)\right) dt$$

which gives

$$\begin{aligned}\delta S &= \epsilon \int \left(p\dot{\eta} + \chi\dot{q} - \frac{\partial H}{\partial q}\eta - \frac{\partial H}{\partial p}\chi) + \mathcal{O}(\epsilon^2)\right) dt \\ &= \epsilon \int_{t_1}^{t_2} dt \left((\dot{q} - \frac{\partial H}{\partial p})\chi - (\dot{p} + \frac{\partial H}{\partial q})\eta\right)\end{aligned}$$

In order for this to go to zero, we must have

$$\dot{q} - \frac{\partial H}{\partial p} = 0 \qquad \dot{p} + \frac{\partial H}{\partial q} = 0$$

which derives the equations of motion!

9.7 Ray Tracing

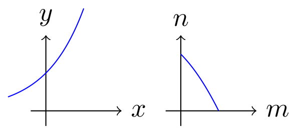
We can analogize this to ray tracing (which if I had to guess won't be on the exam). Basically, we can use energy to follow paths through ray tracing.

10 Canonical Transformation

10.1 Laying the Groundwork

10.1.1 Legendre Transformations

Consider some function



We're transforming to tangency space, so at some point we have (in two variables)

$$f(x,y)$$

so that

$$f(x,y) \to df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

and write

$$u = \frac{\partial f}{\partial x}$$
$$v = \frac{\partial f}{\partial y}$$

so we want to transform from $(x,y) \to (v,y)$. write that

$$g = f - ux$$
$$dg = df - udx - xdu = (udx + vdy) - udx - xdu$$
$$dg = vdy - xdu = \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial v}du$$

which, post legendre-transform gives

$$v = \frac{\partial g}{\partial y} \qquad x = -\frac{\partial g}{\partial u}$$

which kind of looks like the hamiltonian!

10.1.2 Hamiltonian⇔Legendre Transformation

$$(q, \dot{q}, t) \to (q, p, t)$$

We can show this by saying that (letting lagrangian correspond now to L instead of $\mathcal{L}$),

$$dL = \frac{\partial L}{\partial q}dq + \frac{\partial L}{\partial \dot{q}}d\dot{q} + \frac{\partial L}{\partial t}dt$$
$$p = \frac{\partial L}{\partial \dot{q}}$$
$$\Downarrow$$
$$dL = \dot{p}dq + p\dot{q} + \frac{\partial L}{\partial t}dt$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} = \dot{p}$$

This then becomes

$$dH = \dot{q}dp + pd\dot{q} - \left(\dot{p}dq + pd\dot{q} + \frac{\mathrm{d}L}{\mathrm{d}t}dt\right)$$
$$= \dot{d}p - \dot{p}dq - \frac{\partial L}{\partial t}dt = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial t}dt$$

which, equating term by term gives hamiltons equations of motion, under a legendre transformation.

10.1.3 Steps for Solving Hamiltonian Dynamics

1. Choose coordinates q, construct $\mathcal{L}$.
2. Get Canonical Momentum $p = \frac{\partial \mathcal{L}}{\partial q}$.
3. Find the hamiltonian $H(q, \dot{q}, p, t) = \dot{q}p - \mathcal{L}(q, \dot{q}, t)$.
4. invert $p = \frac{\partial \mathcal{L}}{\partial q}$ to get $\dot{q}(q, p, t)$
5. Eliminate $\dot{q}$ from $H \to H(q, p, t)$.

10.1.4 Particle in Gravity

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

and

$$p_x = m\dot{x} \qquad p_y = m\dot{y} \qquad p_z = m\dot{z}$$

which gives

$$H = \frac{1}{m}\left(p_x^2 + p_y^2 + p_z^2\right) - \left(\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz\right)$$

Under the Legendre transformation, we get that

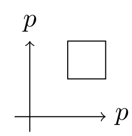
$$H(q,p) = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + mgz$$

which gives that $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$, so the whole thing just comes out to be that

$$\dot{p}_z = p\frac{\partial H}{\partial q} = -mg$$

10.1.5 Liouville Theorem

Let's take phase space



Let some box in p, q phase space with side lengths $\Delta p, \Delta q$. Let $\rho \equiv$density of points in q, p, or phase space density.

How many particles cross the left face of the box (closest the p axis, orthogonal to q) in time dt?

We write $dn = \rho \Delta q \Delta p$, with $\Delta q = \frac{\mathrm{d}q}{\mathrm{d}t}dt = qdt$ With indices, we can say that

$$dN_2 = \rho\dot{q}\big|_{q+\Delta q}dt\Delta p$$

which gives

$$dN_{12} = dN_1 - dN_2 = \left(\rho\dot{q}\big|_q - \rho\dot{q}\big|_{q+\delta q}\right)dt\Delta p$$

we can do the same thing along the p axis as well.

Interpreting this, we can think of particles entering and exiting the box. If more go in than come out, then we have some weird shenanigans happening.

We can write the p, q equations as derived above as a single differential equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q}(\rho\dot{q}) + \frac{\partial}{\partial p}(\rho\dot{p}) = 0$$

which is a continuity equation for phase space.

Simplifies to

$$\left(-\frac{\partial}{\partial q}(\rho\dot{q}) - \frac{\partial}{\partial p}(\rho\dot{p})\right)dt\Delta q\Delta p = \frac{\partial \rho}{\partial t}\Delta t\Delta q\Delta p$$

where the left side is the particles entering, exiting the box in each direction, and the right side is the change in density.

Wrriten in hamiltonian mechanics, the liouville theorem is expressed as

$$\frac{\partial p}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial \rho}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial \rho}{\partial p} = 0$$

watch out for the Poisson Brachet. It'll come up later.

10.2 Canonical Transformations (not in book)

There's some discussion of this in Hand + Finch.

We have some $H(q,p,t)$, where p is the canonical momentum.

$$\dot{p} = -\frac{\partial H}{\partial q}$$
$$\dot{q} = \frac{\partial H}{\partial p}$$

If we have a cyclic coordinate, it simplifies the problem, i.e. if $\frac{\partial H}{\partial q} = 0$, then $\dot{p} = 0 \Rightarrow p$ is a constant.

Good example of this principle is central force problems, since $\mathcal{L} = \frac{1}{2}m\dot{r}^2 + r^2\dot{\theta}^2 - V(r)$, potential exclusively depends on r, which means $\frac{\partial L}{\partial \theta} = 0 \to l = mr^2\dot{\theta} \equiv$ constant.

Canonical Transformations are ways of finding convenient coordinates to make the hamiltonian cyclic as well.

Lets call the transform $Q = Q(p_i, q_i, t)$, and $P = P(q_i, p_i, t)$.

If it makes some $\mathcal{H}$ cyclic in Q, then $\mathcal{H} = \mathcal{H}(P)$, which implies $\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = \omega \equiv$ constant, $Q = \omega t + Q_0$, $\dot{p} = -\frac{\partial \mathcal{H}}{\partial Q} = 0$, $p \equiv$const.

(q,p) are called canonically conjugate, that if Hamiltons equations hold for $(q,p) \Leftrightarrow (Q,P)$ under canonical equations.

Hamiltons Principle

$$\delta \int L(q,\dot{q},t)dt = 0 = \delta \int \mathcal{L}(Q,\dot{Q},t)dt$$

this means that $\delta(L - \mathcal{L})dt = 0$. Because this is a time integral, the lagrangian and the transformed lagrangian can differ by a total differential and this would still be true.

$$\delta \int_{t_1}^{t_2} \frac{\mathrm{d}F}{\mathrm{d}t}dt = \delta\left(F(t_2) - F(t_1)\right) = 0$$

which implies that

$$L - \mathcal{L} = \frac{\mathrm{d}F}{\mathrm{d}t}$$

with F called the **generating function**. It should have $2n+1$ independent variables.

There are four types of generating functions

$$F_1 = F(q_i, Q_i, t)$$
$$F_2 = F(q_i, P_i, t)$$
$$F_3 = F(p_i, Q_i, t)$$
$$F_4 = F(p_i, P_i, t)$$

going back to transformed lagrangian, we take $L = \mathcal{K} + \frac{\mathrm{d}F}{\mathrm{d}t} = \sum p\dot{q} - H$, we can write this as

$$\sum p\dot{q} - H = \sum P\dot{Q} - \mathcal{H} + \frac{\mathrm{d}F}{\mathrm{d}t}$$

10.2.1 Type 1 Generator

This gives that

$$\frac{\mathrm{d}F_1}{\mathrm{d}t} = \frac{\partial F_1}{\partial q}\dot{q} + \frac{\partial F_1}{\partial Q}\dot{Q} + \frac{\partial F_1}{\partial t}$$

This can be expressed as

$$\sum p\dot{q} - \sum P\dot{Q} - H + \mathcal{H} = \sum \frac{\partial F_1}{\partial q}\dot{q} + \sum \frac{\partial F_1}{\partial Q}\dot{Q} + \frac{\partial F_1}{\partial t}$$

which gives that

$$p_i = \frac{\partial F_1}{\partial q_i}$$
$$P_i = -\frac{\partial F_1}{\partial Q}$$
$$\mathcal{H} = H + \frac{\partial F_1}{\partial t}$$

10.2.2 2 Examples

Coordinate Swap take $F_1(q,Q) = qQ$. Then

$$p_i = \frac{\partial F}{\partial q} = Q$$
$$P_i = -\frac{\partial F}{\partial Q} = -q$$

Simple Harmonic Oscillator Recall $L = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2$, which gives

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \to q = \frac{p}{m}$$
$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$$
$$\omega^2 = \frac{k}{m}$$

Let's try this:

$$p = f(P)\cos(Q)$$
$$q = \frac{f(P)}{m\omega}\sin(Q)$$

if we put these into the old hamiltonian, we get

$$p^2 + m^2\omega^2 q^2 = f(P)^2\cos^2(Q) + f(P)^2\sin^2(Q) = f(P)^2$$

which gives our new hamiltonian as

$$\mathcal{H} = \frac{f(P)^2}{2m}$$

Let's get a new type one generator as defined above, so that

$$p = \frac{\partial F}{\partial q}$$
$$P = -\frac{\partial F}{\partial Q}$$

Carrying on with the hamiltonian that we already have, we get

$$p = m\omega q \cot Q = \frac{\partial F}{\partial q}$$

which gives that

$$F = \int p dq = \frac{1}{2}m\omega q^2 \cot Q$$

it is also easy to see that $p = -\frac{\partial F}{\partial Q} = \frac{1}{2}m\omega^2 q^2 \frac{1}{\sin^2(Q)}$,

$$q = \sqrt{\frac{2P}{m\omega}}\sin(Q)$$
$$p = m\omega\sqrt{\frac{2P}{m\omega}}\cos(Q)$$

putting this all into the hamiltonian, we have

$$H = \frac{1}{2m}\left[2Pm\omega\cos^2 Q + m\omega 2P\sin^2 Q\right]$$

which gives, cancelling out, that

$$\mathcal{H} = \omega P$$

since we knnow it doesn't depend on time, we can write that

$$E \equiv \mathcal{H} = \omega P$$

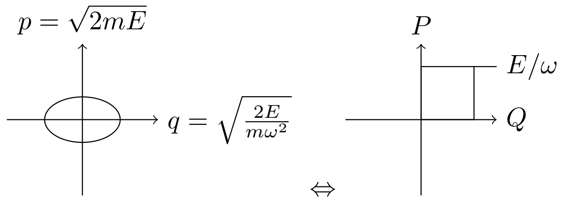
so $P = \frac{E}{\omega}$, or energy per unit angular momentum, so we have that $\dot{Q} = \omega$, which gives

$$Q = \omega t + Q_0$$

Putting these back into the original solution, we find that

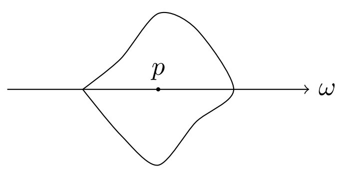
$$p = \sqrt{2mE}\cos(\omega t + Q_0)$$
$$q = \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + Q_0)$$

Let's look at phase space!



11 Rigid Body Motion

Definition 11.1. A rigid body is a body in which the mass elements are fixed with respect to one another.



We have $\vec{L} = \vec{r} \times \vec{p}$, and $p = m\omega$, with point p rotating at an angle θ away from ω, and momentum p at $r' = r\sin\theta$ along that vector

Some mass element δm, we get $\delta\vec{L} = \vec{r} \times \delta\vec{p} = \vec{r} \times \vec{v}\delta m$, which gives

$$\vec{L} = \int \delta m(\vec{r} \times (\vec{\omega} \times \vec{r})$$

or for discrete mass elements, we have

$$\vec{L} = \sum_i m_i \vec{r}_i \times (\omega \times \vec{r}_i)$$

Then, we actually do the cross product out, we get

$$\vec{\omega} \times \vec{r} = (\omega_2 z - \omega_3 y)\hat{x} + (\omega_3 x - \omega_1 z)\hat{y} + (\omega_1 y - \omega_2 x)\hat{z}$$

, so the whole thing comes out to be, after crossing with r again,

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} \int (y^2+z^2)dm & -\int xydm & -\int zxdm \\ -\int xydm & \int (z^2+x^2)dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int (x^2+y^2)dm \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_4 \end{bmatrix}$$

with that big ol matrix defined as the *Inertia Tensor*, $\vec{I}$.

Inertia Tensor has a couple of properties

- Symmetric and Positive Definite.
- Depends only on the *shape* of the system, not ω.
- Can only be calculated after choosing an origin and coordinate system.
- Is diagnoalizable.

We could also write it in the following way,

$$I_{ij} = \int_{\text{all } V} \rho(\vec{r}) \left(\delta_{ij} \sum_k (x_k)^2 - x_i x_j \right) dV$$

11.1 ex: Point mass in a plane

Some mass orbiting $\hat{z}$ in the $x - y$ plane, m, wiht $\omega = (0, 0, \omega_3)$ and $x^2 + y^2 = r^2$, we have

$$\vec{L} = \begin{bmatrix} \int y^2 & -\int xy & 0 \\ -\int xy & \int x^2 & 0 \\ 0 & 0 & \int (x^2+y^2)dm \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \omega_3 \int (x^2+y^2)dm \end{bmatrix}$$

which just reduces to $\vec{L} = mr^2\omega_3\hat{z} = mvr\hat{z}$, which is what we expected anyways.

There's also the parallel axis theorem, which we will discuss later.

11.2 Kinetic Energy

We can also examine the kinetic energy, which is given by

$$dT = m\frac{v^2}{2} = \frac{dm|\vec{\omega} \times r|^2}{2}$$
$$T = \frac{1}{2}\int \left((\omega_2 z - \omega_3 y)^2 + (\omega_3 x - \omega_1 z)^2 + (\omega_1 y + \omega_3 x)^2 \right) dm$$
$$= \frac{1}{2}\vec{\omega} \cdot (\vec{I} \cdot \vec{\omega}) = \frac{1}{2}\vec{\omega} \cdot \vec{L}$$

11.3 Center of Mass Coordinates

Say $r = \vec{R} + \vec{r}'$, where $\vec{R}$ goes from origin to center of mass.

we have

$$\vec{L} = \int dm(\vec{r} \times \vec{v}) = \int \left((\vec{R} + \vec{r}') \times (\vec{r}' + (\vec{\omega} \times \vec{r}')) \right)$$
$$(\vec{R} + \vec{r}') \times (\vec{V} + (\vec{\omega} \times \vec{r}') = \vec{r}' \times \vec{v}' + \vec{r}' \times (\vec{\omega} \times \vec{r}')$$

which gives that

$$\vec{L} = m\vec{R} \times \vec{V} + \vec{L}_{cm}$$

We can also do this for KE, which would give us that

$$T = \frac{1}{2}\int dmV^2 = \frac{1}{2}\int dm|\vec{V} + (\vec{\omega}' \times \vec{r}')|^2$$
$$= \frac{1}{2}MV^2 + \frac{1}{2}\vec{\omega}' \cdot \vec{L}_{cm}$$

11.4 Principal Axes

Goal is to diagonalize the inertial tensor.

$$\vec{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

where I_1, I_2, I_3 are defined as the principal moments.

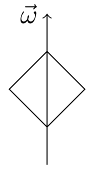
This kind of thing should be somewhat familiar from freshman mechanics classes, since we have something like a plate in $\mathbb{R}^3$, orthogonal to the z axis, the cross terms like $xydm$ cancel in the inertia tensor, but you get and eigenvalue problem.

We want $\vec{L} = \vec{I} \cdot \vec{\omega_1} = I_1\omega_1$, or ω_1 lies along a principal axis.

Want

$$\det \left| \begin{bmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} - I & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} - I \end{bmatrix} \right| = 0$$

11.4.1 Todo next lecture



11.5 Inertial Tensor

$$\Omega_1 = \sqrt{\left(\frac{I_2 - I_1}{I_2}\right)\left(\frac{I_2 - I_1}{I_1}\right)}\omega_2$$

cyclic differential equation

$$\dot{\omega}_1 + \left(\frac{I_3 - I_2}{I_1}\right)\omega_2\omega_3 = 0$$
$$\dot{\omega}_2 + \left(\frac{I_1 - I_3}{I_2}\right)\omega_1\omega_3 = 0$$
$$\dot{\omega}_3 + \left(\frac{I_2 - I_1}{I_3}\right)\omega_2\omega_1 = 0$$

11.6 Spinning Top

If we want to be more rigorous, we should include gravity and torque.

Let some coordinate system $\vec{r}' \equiv$ space coordinates $\equiv$ fixed in space.

$\vec{r}$ is our body coordinates, rotating with some spinning top.

These are related by $\vec{r}' = U\vec{r}$.

Also θ, φ, ψ are the euler angles. We need to choose a convenient definition for these, so let θ be the angle between $z, \hat{x}'_3$, and φ the angle betwween $x, \hat{x}'_2$, and ψ the angle between the xy plane and $\hat{x}'_2$.

So, step 1 is that if we rotate around $z - \hat{x}_3$ by some angle φ (in xy plane), we get new coordinate transformation $\alpha, \beta\gamma$, which gives

$$\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} = \begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

11.6.1 Step 2

Rotate around α by angle θ. Su, we get

$$\begin{bmatrix}\alpha'\\ \beta'\\ \gamma'\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}$$

11.6.2 Step 3

Finally, wrotate by ψ about γ'

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}$$

11.7 gettting the euler angles! (space 2 body)

$U^* = U_3^* U_2^* U_1^*$ transforms from space to body, with $U = U_1 U_2 U_3$ transforms from body to space.

We get the big transpose form of U as

$$U^* = \begin{bmatrix}\cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta & \cos\psi\sin\varphi + \sin\psi\cos\varphi\cos\theta & \sin\theta\sin\psi\\ -\sin\psi\cos\varphi - \cos\psi\sin\varphi\cos\theta & -\sin\varphi\sin\psi + \cos\psi\cos\varphi\cos\theta & \sin\theta\cos\psi\\ \sin\theta\sin\varphi & -\cos\varphi\sin\theta & \cos\theta\end{bmatrix}$$

11.8 Euler Angles, Body/Space Frames

The beginning of lecture was some complicated example using the euler angles to transform into body coordinates. Here's another version of this problem.

11.8.1 Symmetric Top

We have $I_1 = I_2 = I$. It can be a cube, or really something arbitrary. I_3 might be different.

KE can be written $T = \frac{1}{2}\vec{\omega}(I \cdot \vec{\omega}) = \frac{1}{2}(I(\omega_1^2 + \omega_2^2) + I_3\omega_3^2)$.

Using this, and the formulation of the euler angles, we can write down htat

$$\omega_1^2 = \dot{\theta}^2\cos^2\chi + \dot{\varphi}^2\sin^2\chi\sin^2\theta + 2\dot{\theta}\dot{\varphi}\cos\chi\sin\chi\sin\theta$$

with omega 2 defined in a similar manner, which gives

$$\omega_1^2 + \omega_2^2 = \dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta$$
$$\omega_3^2 = (\dot{\chi} + \dot{\varphi}\cos\theta)^2$$

In gravity, there's also potential $V = mgl\cos\theta$, so we can write down the lagrangian

$$\mathcal{L} = \frac{I}{2}(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{I_3}{2}(\dot{\chi} + \dot{\varphi}\cos\theta)^2 - mgl\cos\theta$$

From our work on canonical variables, we can note that

$$p_\varphi = \text{constant}$$
$$p_\chi = \text{constant}$$

we can calculate these then, as

$$p_\chi = \frac{\partial\mathcal{L}}{\partial\dot{\chi}} = I_3(\dot{\chi} + \dot{\varphi}\cos\theta) = I_3\omega_3$$

Which is angular momentum around x_3.

Also, define $a = \frac{I_3\omega_3}{I} \equiv$const.

Now, we can calculate

$$p_\varphi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = I\dot{\varphi}\sin^2\theta + I_3(\dot{\chi} + \dot{\varphi}\cos\theta)\cos\theta \equiv \text{constant}$$

Define $b = \frac{p_\varphi}{I}$.

We can calculate the total energy then as

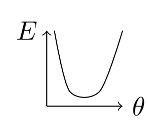
$$E = T + U = \frac{I_2}{2}(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{I_3}{2}\omega_3^2 + Mgl\cos\theta$$

This whole equation simplifies down to a nice form, which is $b = \dot{\varphi}\sin^2\theta + a\cos\theta$.

If we wirte $E' = E - \frac{I_3\omega_3^2}{2}$, then we can have a one-dimensoinal equation

$$E' = \frac{I}{2}\dot{\theta}^2 + \frac{I}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + mgl\cos\theta$$

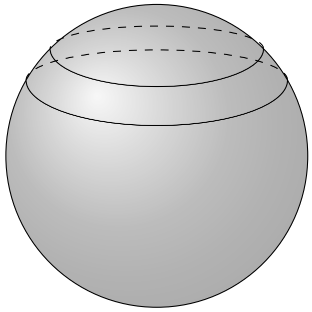
We can look at this like an energy equation



If we define $\dot{\theta}^2(1 - u^2) = \dot{u}^2$, we can do some fancy substitution.

As $u \to \infty$, everythign goes as u^3. Likewise for $-\infty$.

If we now consider $\dot{\varphi} = \frac{b - a\cos\theta}{\sin^2\theta}$, we can see that it will not have zero average. In fact, it will look like some curly path between these two lines on a sphere.



12 Small Oscillations

12.1 Stationary Points

Consider a lagrangian $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x)$, with generalized coordinates $(x, \dot{x})$.

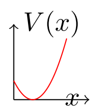
Recall ELE

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

a stationary point is a point x_0 such that $\dot{x}_0 = 0 \Rightarrow = x_0 \forall t$, which gives that $\frac{\partial V}{\partial x} = 0$.

It's a point with no force acting on a system.

mass on spring A mass on a spring (simple harmonic oscillator) has potential $V(x) = \frac{1}{2}k(x - x_*)^2$, which looks as



at those stationary points, there are small oscillations that we can taylor expand to have that

$$m\delta\ddot{x} = -\frac{\mathrm{d}^2V}{\mathrm{d}x^2}_{x=x_i}\delta x$$
$$\delta x = Ae^{-i\omega t}$$
$$\omega^2 = \frac{1}{m}\frac{\mathrm{d}^2V}{\mathrm{d}x^2}_{x=x_i}$$

The most general expression for a lagrangian is

$$\mathcal{L}(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

with

$$T = \frac{1}{2}\sum T_{ik}\dot{q}_i\dot{q}_k$$

or, in other words

$$T = \frac{1}{2}\sum_i m_i|\dot{\vec{r}_i}|^2$$

which allows some reexpression of T as

$$T = \frac{1}{2}\sum_{i,\alpha\beta} m_i \frac{\partial \vec{r}_i}{\partial q_\alpha}\frac{\partial \vec{r}_i}{\partial q_\beta}\dot{q}_\alpha\dot{q}_\beta$$

$$= \frac{1}{2}\sum_{\alpha,\beta}\left[\sum_i m_i \frac{\partial \vec{r}_i}{\partial q_\alpha}\frac{\partial \vec{r}_i}{\partial q_\beta}\right]\dot{q}_\alpha\dot{q}_\beta$$

if we wanted to write this down as a symmetric tensor ($T_{ij} = T_{ji}$), then we should take some kinetic energy of the form

$$T = \frac{1}{2}(T_{11}\dot{q}_1^2 + T_{12}\dot{q}_1\dot{q}_2 + T_{21}\dot{q}_1\dot{q}_2 + T_{2@}\dot{q}_2^2)$$

and reexpress it as

$$T = \frac{1}{2}(T_{11}\dot{q}_1^2 + \frac{T_{12} + T_{21}}{2}\dot{q}_1\dot{q}_2 + \frac{T_{12} + T21}{2}\dot{q}_2\dot{q}_1 + T_{22}\dot{q}_2^2)$$

For a more abstrat system, for a stationary point q^0 where the superscript denotes stationary status, not exponentiation, we have that $\dot{q}_k = 0$ for ever k index, and that $\frac{\partial V}{\partial q_\alpha} = 0 \forall \alpha \in$ our range.

For the case where T_{ik} depends on q, then our lagrangian is

$$\mathcal{L} = \frac{1}{2}\sum_{i,k} T_{ik}(q)\dot{q}_i\dot{q}_k - V(q)$$

which we can apply standard ops to to derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_k T_{\alpha k}\dot{q}_k\right] = \frac{1}{2}\sum_{i,k}\frac{\partial T_{ik}}{\partial q_\alpha}\dot{q}_i\dot{q}_k - \frac{\partial V}{\partial q_\alpha}$$

the final form comes out to be

$$\sum_k T_{\alpha k}\ddot{q}_k = -\frac{\partial V}{\partial q_\alpha} + \frac{1}{2}\sum_{i,k}\frac{\partial T_{ik}}{\partial q_\alpha}\dot{q}_i\dot{q}_k - \sum_{k,s}\frac{\partial T_{\alpha k}}{\partial q_s}\dot{q}_s\dot{q}_k$$

which are newtons equations for this lagrangian.

Now, we are considering the behavior of a system around a stationary point in n generalized coordinates.

We take the lagrangian and expand

$$\mathcal{L} = \frac{1}{2}\sum_{i,k} T_{ik}(q)\dot{q}_i\dot{q}_k - V(q_i)$$

if we expand up to quadratic terms, we are taking

$$\mathcal{L} = \frac{1}{2}\sum_{i,k} T_{ik}(q^{(0)} + \delta q)\delta\dot{q}_i\delta\dot{q}_k - V(q^{(0)} - \sum_i \frac{\partial V}{\partial q_i}_0 \delta q_i - \frac{1}{2}\sum_{i,j}\frac{\partial^2 V_{ij}}{\partial q_i \partial q_j}\delta q_i \delta q$$

Constant terms we set to zero, and we get that (setting the mass tensor $T_{ik}(q^{(0)} = m_{ik}$ and V_{ij} another tensor whos name i forget, we have

$$\mathcal{L} = \frac{1}{2}\sum_{i,k} m_{ik}\dot{q}_i\dot{q}_k - \frac{1}{2}\sum_{ik} V_{ik}q_iq_k$$

in a single equation, we just end up with $m\ddot{x} = -kx$ which is what we expect.

This is apparently pretty easy in particular systems, so let's take a look at an example.

12.1.1 Example: Coupled Pendulum

Consider two identitical masses connected by two identical ropes, ith generalized coordinates ϕ_1, ϕ_2, in a cartesian x, y system.

So,

$$x_1 = e\sin\phi_1 \qquad y_1 = -e\cos\phi_1$$
$$x_2 = e\sin\phi_1 + e\sin(\phi_1 + \phi_2) \qquad y_2 = -e\cos\phi_1 - e\cos(\phi_1 + \phi_2)$$

With the conclusion that

$$T_1 = \frac{1}{2}ml^2\dot{\phi}_1^2$$

and

$$T_2 = \frac{1}{2}m[l^2\dot{\phi}_1^2 + l^2(\dot{\phi}_1 + \dot{\phi}_2)^2 + 2l^2\dot{\phi}_1(\dot{\phi}_1 + \dot{\phi}_2)\cos\phi_2]$$

The total kinetic energy then, is (after a lot of algebraic simplification

$$T = \frac{1}{2}ml^2\left[2\dot{\phi}_1^2 + (\dot{\phi}_1 + \dot{\phi}_2)^2 + 2\dot{\phi}_1(\dot{\phi}_1 + \dot{\phi}_2)\cos\phi_2\right]$$

Potential energy is given by

$$V = -mgl\cos\phi_1 - mg(l\cos\phi_1 + l\cos(\phi_1+\phi_2)$$
$$= -mgl(2\cos\phi_1 + \cos(\phi_1+\phi_2)$$

So we want

$$\frac{\partial V}{\partial \phi_1} = 0 \Rightarrow \sin(\phi_1) + \sin(\phi_1+\phi_2) = 0$$

If we want to standardize our kinetic energy, we should rewrite it as

$$T = \frac{1}{2}ml^2[2\dot\phi_1^2 + \dot\phi_1^2 + 2\dot\phi_1\dot\phi_2 + \dot\phi_2^2 + 2\dot\phi_1^2\cos\phi_2 + 2\dot\phi_1\dot\phi_2\cos\phi_2]$$
$$= \frac{1}{2}ml^2[(3+2\cos\phi_2)\dot\phi_1^2 + \dot\phi_2^2 + 2\dot\phi_1\dot\phi_2(1+\cos\phi_2)]$$

which gives

$$T_{11} = (3+2\cos\phi_2)ml^2$$
$$T_{12} = T_{21} = (1+\cos\phi_2)ml^2$$
$$T_{22} = ml^2$$

Now, we have to expand the system, so that

$$V = -mgl\left[\left(1-\frac{\phi_1^2}{2}\right)2 + \left(1 - \frac{(\phi_1+\phi_2)^2}{2}\right)\right]$$
$$V = \frac{1}{2}mgl[2\phi_1^2 + (\phi_1+\phi_2)^2] = \frac{1}{2}mgl[3\phi_1^2 + 2\phi_1\phi_2 + \phi_2^2]$$

when we ignore constants, which is allowable because of the lagrangian formalism.

Kinetic energy about our expansion goes as

$$\frac{1}{2}ml^2\left[5\dot\phi_1^2 + 4\dot\phi_1\dot\phi_2 + \dot\phi_2^2\right]$$

Finally, this gives us the lagrangian

$$\mathcal{L} = \frac{1}{2}ml^2\left(5\dot\phi_1^2 + 4\dot\phi_1\dot\phi_2 + \dot\phi_2^2\right) - \frac{1}{2}mgl(3\phi_1^2\phi_1\phi_2 + \phi_2^2)$$

which means we can write down

$$m_{ik} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$
$$V_{ik} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\omega_0^2$$

where $\omega_0^2 = g/l$. This just gives us a solution of the form $q_k = A_k e^{-i\omega t}$, which we know how to solve.

$$-\omega^2 m_{\alpha k} + V_{\alpha k})A_k = 0$$

which is a statement about whether or not the solution has nontrivial solutions, i.e. it only does if

$$\det\left(\hat V - \omega^2\hat m\right) = 0$$

Now, let's try taking

$$\sum_{ik} A_i^{(s)}(V_{ik} - \omega_s^2 m_{ik})A_k^{(s)} = 0 \Rightarrow \omega_S^2 = \frac{V_{ik}A_i^{(s)}A_k^{(s)}}{m_{ik}A_i^{(s)}A_k^{(s)}}$$

We now want to solve

$$\det\left(\begin{bmatrix} 3\omega_0^2 - 5\omega^2 & -\omega_0^2 - 2\omega^2 \\ \omega_0^2 - 2\omega^2 & \omega_0^2 - \omega^2 \end{bmatrix}\right)$$

which gives a characterisitc equation

$$\omega^4 - 4\omega^2\omega_0^2 + 2\omega_0^2 = 0$$

which gives that

$$\omega_1^2 = \omega_0^2(2+\sqrt{2})$$

and

$$\omega_2^2 = \omega_0^2(2-\sqrt{2})$$

and then we find the eigenvectors of this matrix using usual linear algebra methods.

Recall we have some $\mathcal{L} = \frac{1}{2}\sum_{i,k} T_{ik}\dot q_i\dot q_k - V(q)$ is the most general lagrangian, with $\sum_{i,k} T_{ik}\eta_i\eta_k > 0 \forall \eta_i, \eta_1^2 + \ldots + \eta_n^2 \neq 0$.

Then, we find stationary point $\frac{\partial V}{\partial q} = 0 \Rightarrow q_i^{(0)}$, $q_2 = q_i^{(0)} + \delta q_i$, which expands to the following form

$$\mathcal{L} = \frac{1}{2}\left(\sum_{i,k} m_{ik}\delta\dot q_i\delta\dot q_k - \sum_{i,k} V_{ik}\delta q_i\delta q_k\right)$$

with $V_{ik} = \frac{\partial^2 V}{\partial q_i\partial q_k}{}_{q^{(0)}}$.

Guess solution is of the form $q_k = A_k e^{-i\omega t}$, or more generally,

$$\sum_k (V_{ik} - \omega^2 m_{ik})A_k = 0$$

or

$$(\hat V - \omega^2\hat m)\vec A = 0$$

12.2 Some Hamiltonian

we're using the einstein summation convention.

$$H = p_j\dot q_j - \mathcal{L} = \frac{\partial\mathcal{L}}{\partial\dot q_j}q_j - \mathcal{L}$$

this expands to

$$\frac{\partial L}{\partial\dot q_j} = \frac{\partial}{\partial\dot q_j}\left[\frac{1}{2}\sum_{i,k} T_{ik}\dot q_i\dot q_k\right] = \frac{1}{2}\left[\sum_{i,k} T_{ik}\delta_{ij}\dot q_k + \sum_{i,k} T_{ik}\dot q_i\delta_{kj}\right]$$

which gives generalized momentum (of further simplification

$$p_j = \sum_i T_{ij}\dot q_i$$

with

$$H = \sum_{i,j} T_{ij}\dot q_i\dot q_j - \frac{1}{2}T_{ij}\dot q_i\dot q_j + V(q) = T + V = \frac{1}{2}\sum T_{ik}\dot q_i\dot q_k + V(q)$$

Recall that T_{ik} a function of q. So, we want to expand the hamiltonian around small oscillations or some stationary point, and it becomes

$$H = \frac{1}{2}\left(\sum_{i,k} m_{ik}\dot q_i\dot q_k + \sum_{i,k} V_{ik}q_iq_k\right)$$

with $V(q^{(0)}) = 0$ by definition.

Theorem 12.1. Suppose that $\forall\eta_i|\eta_1^2 + \ldots + \eta_n^2 > 0$ we have $V_{ik}\eta_i\eta_k \gg 0$, then te stationary point $q^{(0)}$ is stable, then $q_i = A_i e^{-i\omega t}$, and $\omega \in \mathbb{R}$.

Proof. We have

$$H_0 - \frac{1}{2}\sum_{i,k} V_{ik} q_i q_k = \frac{1}{2}\sum_{i,k} m_{ik}\dot{q}_i\dot{q}_k$$

which, we know the sum term on the right is always positive at a stable point (since it goes up), which gives with H_0 constant, that oscillation, with $\omega \in \mathbb{R}$.

CONNOR NOTE: I'm pretty sure that V_{ik} is just the hessian matrix for V.

Let ω_s^2, A_k^s the eigenvalue, vector correspondent to the root ω_s^2. We have then that

$$(V_{ik} - \omega_s^2 m_{ik})A_k^s = 0$$

and left multiply by adjoint of A_k^s to get

$$\sum_{ik} = \sum_{ik} A_i^s V_{ik} A_k^s = \omega_s^2 \sum_{i,k} m_{ik} A_i^{*(s)} A_k^{*(s)}$$

where we;re allowed to change multiplication order only in index notation because they're numbers.

That gives

$$\omega_s^2 = \frac{\sum_{i,k} V_{ik} A_i^{*(s)} A_k^{(s)}}{\sum_{i,k} m_i k A_i^{*(s)} A_k^{(s)}}$$

want to show that $\forall \eta_i \in \mathbb{C} \sum_{i,k} m_{ik}\eta_i\eta_k > 0$, multiplying the two by complex conjugates makes it poisitive.

Let's introduce another $A_i^{(\alpha)}$, and take

$$A_i^{*(\alpha)} A_k^{(s)} V_{ik} = \omega_s^2 m_{ik} A_i^{*(\alpha)} A_k^{(s)}$$
$$A_i^{(s)} A_k^{*(\alpha)} V_{ik} = \omega_\alpha^2 m_{ik} A_k^{*(\alpha)} A_i^{(s)}$$

which subtracting one from the other

$$\omega_s^2 - \omega_\alpha^2 = m_i k A_i^{(\alpha)} A_k^{(s)} \equiv 0$$

for nondegenerate cases, eigenvectors must satisfy

$$\sum_{i,k} m_{ik} A_i^{*(\alpha)} A_k^{(s)} \equiv 0$$

i.e. for nondegeneracy, it's some sort of generalized norm over a space with metric m_{ik}.

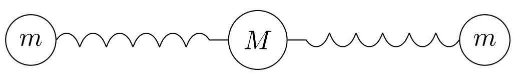
Check out Goldstein for a proof f this.

Finally, suppose we have every frequency, eigenvector, how do we write a general solution?

$$q_k = \textstyle\sum_s C_s A_k^{(s)} e^{-i\omega_s t} \quad C_s \in \mathbb{C}$$

12.2.1 Model of a Molecule

consider some



which gives (after writing down the lagrangian)

$$m_{ik} = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}$$
$$V_{ik} = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

which we diagonalzie to $-k^2(k - \omega^2 m) = 0$, which gives three possible roots

$$\omega = 0$$
$$\omega^2 = \frac{k}{m}$$
$$\omega^2 = \frac{k}{m}(1 + 2\frac{m}{M})$$

we just find the eigenvectors that correspond to these eigenvalues, and we're sitting pretty.

For $\omega = 0$, the velocities are symmetric, potential is independent of direction, only depends on displacement of x_1, x_2, x_3.

for $\omega^2 = \frac{k}{m}$, we get

$$\begin{bmatrix} 0 & -k & 0 \\ -k & 2k - k\frac{M}{m} & -k \\ 0 & -k & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 0$$

which has, so we have $A_2 = 0$, with eigenvector

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

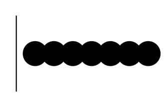
Final eigenfrequency is going to give

$$\begin{bmatrix} -2k\frac{m}{M} & -k & 0 \\ x & x & x \\ 0 & -k & -2k\frac{m}{M} \end{bmatrix}$$

which gives $A_1 = A_3$ and $A_2 = -2\frac{m}{M}A_1$, so eigenvector is

$$A\begin{bmatrix} 1 \\ -2\frac{m}{M} \\ 1 \end{bmatrix}$$

12.2.2 Another Example. More Algebra!



basically, it's a bunch of masses on a string that all are oscillating. It's a bit odd. Work this out, there's #TOO #MUCH #ALGEBRA.

12.3 Something New

Considedr N masses connected by some medium with tension τ, whihcm eans we need y_n $n = 1 \to N$, withh yeach y_k given as the displacement above equilibrium

Lagrangian given as

$$\mathcal{L} = \frac{1}{2}\left(\sum_{k=1}^{N} m\dot{y}_k^2 - \sum_{k=0}^{n}\frac{\tau}{d}(y_k - y_{k+1})^2\right) : y_0 = 0, y_{N+1} = 0$$

we need to assume tht $m_{ik} = m\delta_{ik}$, and we have that

$$V_{ik} = \frac{\tau}{d}\begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots \\ 0 & -1 & 2 & -1 & \dots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & -1 & 2 \end{bmatrix}$$

So, if we want to find the eigenmodes, we just take $||\hat{V} - \omega^2 \hat{m}|| = 0$, which gives us

$$\det \left|\begin{bmatrix} 2 - \frac{\omega^2}{\omega_0^2} & -1 & \dots & 0 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 2 - \frac{\omega^2}{\omega_0^2} \end{bmatrix}\right|$$

If we wanted to solve this for two masses, we have

$$\det \begin{vmatrix} x & -1 \\ -1 & x \end{vmatrix}$$

in 3 dimensions, we should have

$$\det \begin{vmatrix} x & -1 & 0 \\ -1 & x & -1 \\ 0 & -1 & x \end{vmatrix}$$

which gives eigenfrequencies $\omega = \omega_0\sqrt{2}$ and $\omega = \omega_0\sqrt{2 \pm \sqrt{2}}$.

Also always true that $\omega \not\to \infty$. You can just use a computer to do this, but we can use recursive relations.

Lets call Λ_N the determinant of our giant matrix of said form. You can use a recurrence relation $\Lambda_N = x\Lambda_{N-1} - \Lambda_{N-2}$.

If we write down lagranges equation, we get

$$m\ddot{y}_k = \frac{\partial \mathcal{L}}{\partial y_k} = -\frac{\tau}{d}(y_k - y_{k+1}) - \frac{\tau}{d}(y_k - y_{k-1})$$

if we guess teh form is of $y_k = e^{i(k\gamma+\delta)}$, we can reqrite the above as

$$\omega^2 e^{ik\gamma} = \omega_0^2 e^{ik\gamma}(1 - e^{i\gamma}) + \omega_0^2 e^{ik\gamma}(1 - e^{-i\gamma})$$

which can simplify do

$$\omega^2 = \omega_0^2(2 - 2\cos\gamma) = 4\omega_0^2 \sin^2 \frac{\gamma}{2}$$

of course γ not arbitrary, because we must have $y_0 = y_{N+1} = 0$, which means we must have

$$y_k = \cos(ik\gamma + \delta)\cos(\omega t + \varphi)$$

which givess $y_0 = \cos\delta\cos(\omega t + \varphi) \Rightarrow \delta = \frac{(2n+1)\pi}{2}$. and $\gamma = \frac{\pi n}{N+1}$, which just gives out standing waves in the end!

We can think of γ/d as the wavenumber, which, if we think about $\frac{2\pi}{\lambda} = \frac{\gamma}{d}$, our wavenumber is not arbitrary, which means we need to have that equal $\frac{\pi n}{d(N+1)}$, where the denominator is L, the length between boundaries of the medium.

Or, $\frac{L}{\lambda} = \frac{n}{2}$.

12.4 Traveling Wave

This is *not a general solution* however. Imagine the case where we perturb one mass in the center of hundreds of masses. γ is not fixed here, because the boundary conditions don't know about the pertubation until later.

We can try to find a solution for some traveling wave $y_k = A_k e^{i(k\gamma - \omega t)}$, where there's some dispersion relation $\omega = 2\omega_0 \sin\frac{\gamma}{2}$. If we assume small γ, we have immediately that $\omega = \omega_0\gamma = \omega_0 d\vec{k}$ where $\vec{k}$ is the wavevector, with some speed of sound $c_S = \sqrt{\frac{\tau}{md}} \times d = \sqrt{\frac{\tau d}{m}}$.

We can be more strict though. Let's consider the continuous limit of our system, letting $d \to 0, k \to \infty, m \to 0$, with linear mass density $\rho = \frac{m}{d}$.

Now, we have some function $y(x,t)$, with

$$\ddot{y} = \omega_0^2 \frac{\partial y_{k+1/2}}{\partial x} d - \omega_0^2 \frac{\partial y_{k-1/2}}{\partial x} d = \omega_0^2 d^2 \frac{\partial^2 y}{\partial x^2}$$

which is just the wave equation, for constant density, linear mass density.

$$\frac{\partial^2 y}{\partial t^2} = \omega_0^2 d^2 \frac{\partial^2 y}{\partial x^2}$$

Going through the derivation again, we have the more general form of

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{\rho(x)} \frac{\partial}{\partial x}\left[\tau(x)\frac{\partial y}{\partial x}\right]$$

If we rewrite the wave equation as

$$\frac{\partial^2 y}{\partial t^2} = c_s^2 \frac{\partial^2 y}{\partial x^2}$$

we find that in this wave equation, $\frac{\omega}{c_s} L = \pi n$, which is just a large limit of the formula we had before.

12.5 Lagrangian Density

We can also do this using the lagrangian, by making an argument

$$\mathcal{L} = \frac{1}{2}\rho(x)\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\tau(x)\left(\frac{\partial y}{\partial x}\right)^2$$

The way we arrive at this conclusion is by varying the functional

$$\delta \int_{t_1}^{t_2} \int_{\mathcal{D}} \mathcal{L}(x, t, y, \partial_x y, \partial_t y) = 0$$

A WHOLE LOT OF ALGEBRA LATER, the correct answer will fall outof the thing.

the final form, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t y)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x y)}\right) = \frac{\partial \mathcal{L}}{\partial y}$$

12.6 Hamiltonian Density

We have some H, we can introduct some continuous medium version of this, with $\mathcal{L}(x, t, y, \partial_t y, \partial_x y)$, with some generalized momenta $\vec{p} = \frac{\partial \mathcal{L}}{\partial(\partial_t y)}$, then introduce hamiltonain denstiy

$$\mathcal{H} = p\partial_t y = \mathcal{L} = \frac{1}{2}\rho(x)\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}\tau(x)\left(\frac{\partial y}{\partial x}\right)^2$$

Lagrangian density for some discrete masses on string, with y_k change in y, and η_k displacement in x, then

$$\mathcal{L} = \frac{1}{2}\rho(x)\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\tau(x)\left(\frac{\partial y}{\partial x}\right)^2 + O(\eta)$$

where $O(\eta)$ higher order terms

Recall our equation of motion

$$\frac{\partial}{\partial x}\left(\tau(x)\frac{\partial y}{\partial x}\right) = \rho(x)\frac{\partial^2 y}{\partial t^2}$$

if we require τ constant, then we get $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c(x)}\frac{\partial^2 y}{\partial t^2}$.

13 Solving the Wave Equation

Two methods of solving

- Computer
- Pertubation Theory (only valid for $\lambda \ll [\frac{\mathrm{d}}{\mathrm{d}x}(\ln(l_s''))]^{-1}$ or $\lambda \ll \frac{1}{\frac{1}{c_s}\frac{\mathrm{d}L_s}{\mathrm{d}x}}$. Often called the WKB approximation. Basically λ is less than the second x derivative of a typical length scale. GOOGLE.

Let's take the following new variables,

$$\xi = x - vt$$
$$\eta = x + vt$$

so that $(x, t) \to (\xi, \eta)$. So, we now have

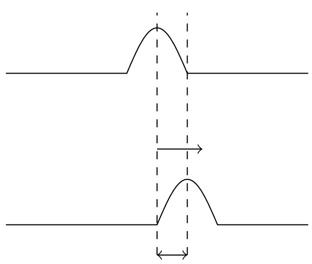
$$\frac{\partial y}{\partial x} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) y$$
$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right) y$$
$$\frac{\partial y}{\partial t} = c_s\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) y$$
$$\frac{\partial^2 y}{\partial t^2} = c_s^2\left(c_s\frac{\partial^2}{\partial \xi^2} - 2\frac{\partial^2}{\partial \eta \partial \xi} + \frac{\partial^2}{\partial \eta^2}\right) y$$

so we can find some solution by setting

$$y(x, t) = f(x - c_s t) + g(x + c_s t)$$

if we set $g \equiv 0$, then

$$y(x, t) = f(x - c_s t) \qquad y(x, 0) = f(x)$$



$c_s \Delta t$

traveling wave solution, it only goes from $x \to x + c_s\Delta t$.

In the arbitrary solution, we have

$$y(x, t) = f(x - ct) + g(x + ct)$$

where f, g are determined by initial conditions. (i.e. at $t = 0$, we have $f(x) + g(x) = y_0(x)$).

taking

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} = \frac{1}{c_s}\dot{y}_0(x)$$

we integrate to see that

$$g(x) - f(x) = \frac{1}{c_s}\int_{x_0}^{x} \dot{y}_0(x')dx'$$

which gives

$$f(x) = y_0(x) = \frac{1}{c}\int_{x_0}^{x} \dot{y}_0(x')dx'$$
$$g(x) = y_0(x) + \frac{1}{c}\int_{x_0}^{x} \dot{y}_0(x')dx'$$

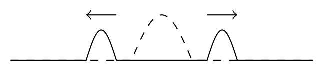
Then, we also have

$$y(0, t) = f(x - ct) + g(x + ct)$$

so we can write the D'lambert solution to the wave equation.

$$y(x, t) = \frac{1}{2}\left[y_0(x - ct) + y_0(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \dot{y}_0(x')dx'$$

for a small pertubation, the solution of this immediately becomes that there are two pulses



propagating in opposite directions.

13.1 General Solution

The generalized wave equation can be written

$$\nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

There are several categories of solutions to consider. First is a standing wave, $\psi(\vec{r}, t)$ depends as $A(\vec{r})e^{-i\omega t}$.

We get the **Helmholtz Equation** that describes such solutions.

$$\nabla^2 A(r) + \frac{\omega^2}{c^2}A(\vec{r}) = 0$$

The 1-d case (say for a string with fixed boundaries is given as

$$\frac{\partial^2 A}{\partial x^2} + \frac{\omega^2}{c^2}A = 0$$

so given that $A(0) = A(L) = 0$, then the boundary condition for the left gives $A(x) = a\sin\left(\frac{\omega}{c}x\right)$. We also need $\sin\left(\frac{\omega}{c}L\right) = 0$, then possible frequencies given $\Omega = \{\frac{c\pi n}{L} | n \in \mathbb{Z}\}$.

So, the general solution is written

$$\psi_n(x, t) = e^{-i\omega_n t}\sin\left(\frac{\pi n}{L}x\right)$$

with $\omega_n = \frac{\pi n}{L}c$.

Fun math fact, we can write down the general solution for one dimension as

$$\psi(x, t) = \sum_{n=1}^{\infty}[a_n\cos(\omega_n)t + b_n\sin(\omega_n t)] \times \sin\frac{\pi n}{L}x$$

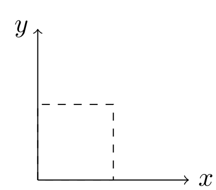
if you do the math out you get thatt

$$a_n = \frac{2}{L}\int_0^L \psi(x, 0)\sin\left(\frac{\pi n}{L}x\right)dx$$
$$\omega_n b_n = \frac{2}{L}\int_0^L \frac{\partial \psi}{\partial t}\sin\frac{\pi}{n}Ldx$$

13.2 2-d Helmholz

$$\psi = e^{-i\omega t}A(x, y)$$

Say some membrane, with ψ some oscillation in z over a bounded membrane



Let's take
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\omega^2}{c^2}\right)A=0$$
under teh assumption that A is separable, i.e. $A=X(x)Y(y)$. Then we can write
$$X''(x)Y(y)+Y''(y)X(x)+\frac{\omega^2}{c^2}XY=0$$
which, we can divide by XY to get
$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{\omega^2}{c^2}=0$$
so, let's write some stuff down
$$\begin{aligned} X''(x)&=-\lambda X, && \lambda\equiv\text{const}\\ Y''(y)&=-\mu Y && \mu\equiv\text{const}\end{aligned}$$
we have the boundary conditions that $X=a\sin\left(\sqrt{\lambda}x\right)$, also with $\sin\left(\sqrt{\lambda}L\right)=0$, so we have
$$\begin{aligned} X_n(x)&=a\sin\left(\frac{\pi n}{L}x\right)\\ Y_m(y)&=b\sin\left(\frac{\pi m}{L}y\right)\end{aligned}$$
we also know that
$$\frac{\omega^2}{c^2}(\lambda+\mu)\Rightarrow\frac{\omega_{nm}^2}{c^2}=\frac{\pi^2(n^2+m^2)}{L^2}$$
which gives some general solution
$$\psi(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_{nm}\cos(\omega_{nm}t)+b_{nm}\sin(\omega_{nm}t))X_n(x)Y_m(y)$$

13.2.1 Circular Boundary

If we have some circular boundary, we still have the same helmholz, and $A=A(r,\theta)$. We rewrite the laplacian in cylindciral coordinates and get
$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\partial A}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 A}{\partial\theta^2}+\frac{\omega^2}{c^2}A=0$$
Apply condition $A(r,0)=A(r,2\pi R_0)$, and $A(r,\theta)=R(r)e^{-im\theta}$.

13.2.2 Bessel function

$$y''(x)+\frac{1}{x}y'(x)+\left(\lambda^2-\frac{n^2}{x^2}\right)y=0$$
so that the *Bessel Functions* of order n are given as solutions to this bad boy.
$$y(x)=J_n(\lambda x)$$
We have
$$R''(r)+\frac{1}{r}R'(r)+\left(\frac{\omega^2}{c^2}-\frac{m^2}{r^2}\right)R=0$$
which gives solution
$$R=J_m(\frac{\omega}{c}r)$$
we apply that it must satisfy the boundary condition $J_m(\frac{\omega}{c_s}R_0=0$, which give solutions. I don't think it's gonna be super important to know how to solve this, but basically it;s the roots of the bessel function (this is the wave equation for a spherically propagating wave, which is how the double slit experiment works!).

so the full on solutions are given as
$$\psi=J_m\left(\frac{\omega_{nm}}{c}\right)\times\begin{Bmatrix}\cos(m\varphi)\\ \sin(m\varphi)\end{Bmatrix}\times\times e^{-i\omega_{nm}t}$$

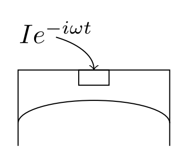
If we consider the case for oscillaating membranes on a cylinder, we'd write down
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^r A}{\partial\theta^r}+\frac{\partial^2 A}{\partial z^2}+\frac{\omega^2}{c^2}A=0$$
which becomes, with $A=R(r)Z(z)e^{-im\theta}$
$$\frac{1}{Rr}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)-\frac{m^2}{r^2}+\frac{Z''(z)}{Z}+\frac{\omega^2}{c^2}=0$$
we just get anotherr
$$Z''(z)+\lambda Z(z)=0$$
which gives solutions of the bessel equation with differend conditions, we fund
$$R''+\frac{1}{r}R'-\frac{m^2}{r^2}R+\left(\frac{\omega^2}{c^2}-\left(\frac{\pi n}{L}\right)^2\right)R=0$$
which reduces
$$R''+\frac{1}{r}R'+\left(\frac{\omega^2}{c^2}-\left(\frac{\pi n}{L}\right)^2-\frac{m^2}{r^2}\right)R=0$$
$$R(r)=J_k(\sqrt{\frac{\omega^2}{c^2}-\left(\frac{\pi n}{L}\right)}r)$$

14 Nonlinear Mechanics + Chaos

You're gonna need to read the chapter and code to solve the majority of these problems. There are only a few of these that we can solve and understand analytically.

14.1 Van der Pol Oscillator

This is a van der pol oscilator.



14.2 Duffing's Oscillator

two springs attached to a mass in a plane with relaxed length l_0. So we have
$$U=\frac{1}{2}k(\sqrt{x^2+l^2}-l_0)^2 2$$

which we expand to be (taylor about 0)

$$\frac{\partial U}{\partial x} = 2k(\sqrt{l^2 + x^2} - l_0)\left(\frac{1}{2}(l^2 + x^2)^{1/2} 2\lambda\right) = 2k\left(x - \frac{x l_0}{\sqrt{l^2 + x^2}}\right)$$

$$\frac{\partial^2 U}{\partial x^2} = 2k\left(1 - \frac{l_0 l^2}{(l^2 + x^2)^{3/2}}\right)$$

$$\frac{\partial^3 U}{\partial x^3} = 6k l_0 l^2 (x^2 + l^2)^{-5/2} x$$

$$\frac{\partial^4 U}{\partial x^4} = \frac{6k l_0}{l^3}$$

So, when we put in the right taylor expansion coefficients, the effective potential beomes

$$U(x) \approx U(0) + k\left(1 - \frac{l_0}{l}\right)x^2 + \frac{1}{4}\frac{k l_0}{l^3}x^4 + \dots$$

So the force becomes

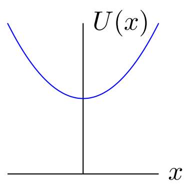
$$F = -\frac{\partial U}{\partial x} = -2k\left(1 - \frac{l_0}{l^2}\right)x - \frac{k l_0}{l^3}x^3$$

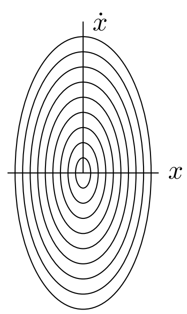
There's a spring term, and a cubic term.

So we want to solve the following differential equation

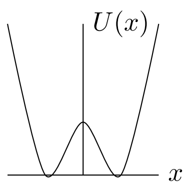
$$m\ddot{x} + 2\beta m\dot{x} + 2k\left(1 - \frac{l_0}{l}\right)x + \frac{k l_0}{l^3}x^3 = f(t)$$

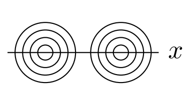
If we check this out in phase space, we look at





if we actially account for the cubic term in potential, however, we're going to get a taylor expansion that looks more like





i.e. there are multiple stable points in phase space.

If we go back to solving this bad boi in generalit, we'll take

$$\ddot{x} + \frac{\dot{x}}{Q} + x + \epsilon x^3 = f\cos\omega t$$

Now, let's let $Q \to \infty$ so there's no damping, and fourier expand this,

$$x(t) = \sum_n A_n(\omega)\cos(n\omega t)$$

with the differential equation being now

$$\ddot{x} + x + \epsilon x^3 = f\cos(\omega t)$$

So now, we want to take a look at the harmonics so we have (even terms go away since they correspond to the sin components))

$$\begin{aligned} x(t) &= A_1\cos(\omega t) + A_3\cos(3\omega t) + \dots \\ \dot{x} &= -A_1\omega\cos\omega t - 3\omega A_3\sin(3\omega t) \\ \ddot{x} &= -\omega^2 A_1\cos\omega t - 9\omega^2 A_3\cos\omega t \end{aligned}$$

Making use of the trig id that

$$\cos^3 x = \frac{3\cos(x) + \cos(3x)}{4}$$

we have our differential equation as

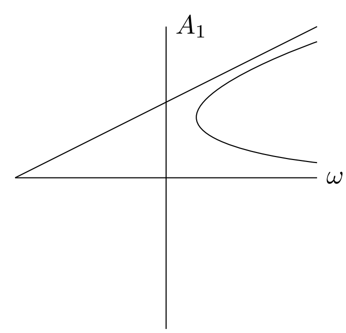
$$\begin{aligned} (1-\omega^2)A_1\cos\omega t + (1 - 9\omega^2)A_3\cos(3\omega t) + \dots \\ + \tfrac{\epsilon}{4}(3A_1^3\cos\omega t + A_1^3\cos(3\omega t) + \dots) \\ = f\cos\omega(t) \end{aligned}$$

if we group our coefficients by n, we get

$$(1-\omega^2)A_1 + \epsilon\frac{3}{4}A_1^3 = f$$

$$(1-9\omega^2)A_4 + \epsilon\frac{1}{4}A_1^3 = 0$$

So, graphically it looks a bit like



where solutions are given as intersections of these curves.

If we put damping back in, we have (i.e. Q finite)

$$\begin{aligned} x &= a\cos\omega t + b\sin\omega t \\ \dot{x} &= -a\omega\sin\omega t + \omega b\cos\omega t \\ \ddot{x} &= -a\omega^2\cos\omega t - b\omega^2\sin\omega t = -\omega^2 x \end{aligned}$$

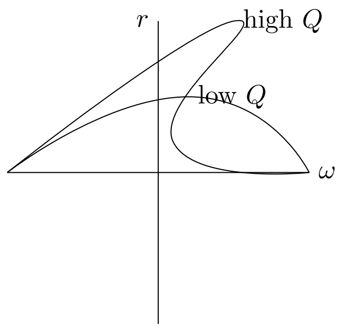
, which gives us an equation from our initial requirements that

$$\begin{aligned} a(1-\omega^2)\cos\omega t + b(1-\omega^2)\sin\omega t + \tfrac{b\omega}{Q}\cos\omega t + \tfrac{3\epsilon a r^2}{4}\cos\omega t \\ + \tfrac{3\epsilon b r^2}{4}\sin\omega t - \tfrac{a\omega}{Q}\sin\omega t \\ = f\cos\omega t \end{aligned}$$

which gives a solution for r^2 as a function of ω

$$r^2 = \frac{f^2}{\left(1 - \omega^2 + \frac{3\epsilon r^2}{4}\right)^2 + \frac{\omega^2}{Q^2}}$$

which gives you pictures that kind of look like this

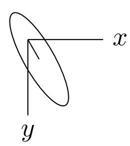


this gives rize to a field of study called catastrophe theory, in which there are discontinuous changes in phase and r.

This is like magnetic fields and hystereisis. Look this up in free time.

14.3 Simple Pendulum

Say we have some rigid body



Small oscillations, we have

$$\hat{r} = r(\hat{y}\cos(\theta+\varphi) + \hat{x}\sin(\theta+\varphi))$$

and

$$\vec{v} = \dot{\theta}r\left[-\hat{y}\sin(\theta+\varphi) + \hat{x}\cos(\theta+\varphi)\right]$$

then for U we write

$$U = mgd(1-\cos\theta)$$

with

$$E = \frac{1}{2}I\dot{\theta}^2 + mgd(1-\cos\theta)\dot{\theta} = \pm\sqrt{\frac{2}{I}(E - mgd(1-\cos\theta)}$$

So we think about $1-\cos\theta = 2\sin^2(\frac{\theta}{2})$. Let θ_0 be the max value of θ, i.e. where our approximation starts to fall apart.

To solve, we can evaluate

$$T = \frac{2r_0}{\sqrt{gd}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}$$

Let $k = \sin\frac{\theta_0}{2}$ and $k_z = \sin\frac{\theta}{2}$, so we write

$$dz = \frac{1}{2}\frac{\cos\theta/2}{k}d\theta$$

which makes

$$T = \frac{2r}{\sqrt{gd}}\int_{z=0}^{1}\frac{zkdz}{\sqrt{1-k^2z^2}}\frac{1}{k\sqrt{1-z^2}}$$

in the small angle approxmation, we have

$$\sqrt{1-k^2z^2} \approx 1 + \frac{1}{2}k^2z^2$$

which allows us to break up T into some shit that requires a lot of trig substitution, which ultimately yields

$$T = \frac{4r_0}{\sqrt{gd}}\left(\frac{\pi}{2} + \frac{1}{2}k^2\frac{\pi}{4}\right) = \frac{2\pi r_0}{\sqrt{gd}}\left(1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + \dots\right)$$

which gives us a term T_0 that tells us how much variation we have from the simple harmonic oscillator

$$T_0 = \frac{2\pi r_0}{\sqrt{gd}}$$

For instance, if we plug in $\theta_0 = 23°$, we get about a 1% change. Now, we do some

14.3.1 Pertubation Theory

and write down the ODE

$$\ddot{x} + \omega_0^2 x - \lambda x^2 = 0$$

which, looking for $x(\lambda, t)$

$$x(\lambda,t) = x_0(t) + \lambda x_1(t) + \lambda^2 x_2(t) + \dots$$

$$\dot{x} = \dot{x}_0 + \lambda\dot{x}_1$$
$$\ddot{x} = \ddot{x}_0 + \lambda\ddot{x}_1$$

which gives

$$\ddot{x} + \omega_0^2 x - \lambda x^2 = 0$$
$$\ddot{x}_0 + \lambda\ddot{x}_1 + \omega_0^2(x_0 + \lambda x_1) - \lambda(x_0 + \lambda x_1)^2 = 0$$
$$\ddot{x}_0 + \lambda\ddot{x}_1 + \lambda\ddot{x}_1 + \omega_0^2\lambda x_1 - \lambda x_0^2 - 2\lambda^2 x_0 x_1 - \lambda^3 x_1 = 0$$

first order perturbation of the thing, so if we're forcing it

$$\ddot{x}_1 + \omega_0^2 = A^2\cos^2\omega_0 t$$

So, what I think he did

$$x(t) = x_0 + \lambda x_1 =_A \cos\omega_0 t - \lambda\frac{A^2}{6\omega_0^2}(\cos(2\omega_0 t) - 3)$$

3-wave coupling.

Kolomogorov (1941). No natural scale/boundary condition in hte problem

Imagine some eddy, that you've just broken up

15 Infinitesimal Canonical Transformations

(this stuff is not examinable) Consider some

$$\tilde{H} = H + \frac{\partial F(g, Q, p, P, t}{\partial t}$$

such that $p_1 = \frac{\partial F}{\partial q}$, $P_1 = -\frac{\partial F}{\partial Q_1}$ where F is some generating function.

Maybe we let $S = qP$, then it's just the identity transformation. If we taek

$$S = qP + \epsilon G(q, p, t)$$

then

$$\tilde{H} = H + \frac{\partial S}{\partial t}$$

and $p = \frac{\partial S}{\partial q}$, and $Q = \frac{\partial S}{\partial P}$, so we want to fund $P(\epsilon) \approx P(\epsilon = 0) + \epsilon\frac{\partial P}{\partial \epsilon}\dots$, and same for Q which gives us

$$0 = \frac{\partial p}{\partial \epsilon} + \frac{\partial G}{\partial q}$$

together, these give

$$P = p - \epsilon \frac{\partial G}{\partial q}$$
$$Q = q + \epsilon \frac{\partial G}{\partial p}$$

Now, if we choose our G to be the hamiltonian, i.e. $G(q, p, t) = H(q, p, t)$, then

$$p \approx p - \epsilon \frac{\partial H}{\partial q} = p + \epsilon \dot{p}$$
$$q \approx q + \epsilon \frac{\partial H}{\partial p} = q + \epsilon \dot{q}$$

If we take ϵ to be dt, then we can think of the hamiltonian as being the propagator of time translation, i.e. using the hamiltonian as a generating function of canonical transformations gives out time translation.

If we take $G = \hat{z} \cdot \hat{L}_z$, we get out that $xp_{-}yp_x$, which takes

$$X \approx x + \epsilon \frac{\partial G}{\partial p_x} = x - \epsilon y$$
$$Y = y + \epsilon \frac{\partial G}{\partial p_y} = y + \epsilon x$$

which shows angular momentum is the generator of rotation!! (Shoutout to my 137a peeps)

What if we want to be more general, examining what happens to a function $u(Q, P, t)$ under such transformations?

$$\left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{t=0} = \left.\left(\frac{\partial u}{\partial Q}\frac{\partial Q}{\partial \epsilon} + \frac{\partial u}{\partial P}\frac{\partial P}{\partial \epsilon}\right)\right|_{\epsilon=0}$$

which is approximately

$$\frac{\partial u}{\partial \epsilon} = \left(\frac{\partial u}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial G}{\partial q}\right)$$

which takes

$$u(t) = u(q, p, t) + \epsilon \left(\frac{\partial u}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial G}{\partial q}\right)$$

where the term on the left is called the Poisson Bracket of $\{u, G\}$

$$\{u, G\} = \frac{\partial u}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial G}{\partial q}$$

Generally, if we take

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{\partial u}{\partial q}\frac{\partial}{\partial q, t} + \frac{\partial u}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial t} \\ &= \frac{\partial u}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial u}{\partial t} \\ &= \{u, H\} + \frac{\partial u}{\partial t}\end{aligned}$$

some more properties of poisson brackets Bale isnt' going to derive.

- $[u, v] = -[v, u]$
- $[u, u] = 0$
- $[(u_1 + u_2), v] = [u_1, v] + [u_2, v]$
- $[u_1 u_2, v] = u_1[u_2, v] + [u_1, v]u_2$
- (also jacobi identity)

and if we have

$$\frac{\partial u}{\partial t} = [u, H] = 0$$

then u is a constant of motion. Also, you can take poisson brackets and generate more conserved quantities (i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}[u, v] = 0$$

16 Final Review, RRR Week

Talking about what bale thinks the priorities are (final is 30%).

Evans 10.

A bit less than comprehensive, thinking 6-7 problems (6.5 on average), 2 of them will be small coupled oscillators, 1 on rigid body rotation, guarunteed.

Others will be distributed amongs lagrangian, linear oscillators and central force motion (probably one of each). Also, Hamiltonian.

You get one x2 sided 8.5x11 cheat sheet. Continuum mechancis/wave equations is not examinable. Think of discrete coupled oscillator problems to begin with. No rutherford scattering.

Rigid body rotation w/ external torques and forces (definitely study this).

Coriolis force, etc not explicitly examinable. Coordinate transformations are really just euler angles

This is just for fun, area is consered under canonical transformations is equivalent to Liouville theorem.

If we talked about generating functions, he would explicitly give us the generator.

Learning more about generating functions

- Griener
- Fetter and Walecka
- Goldstein
- Hand and Finch
- Marion and Thornton
- Symon

¡++¿