Math 222B Notes

Connor Duncan

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Contents

1 Sobolev Spaces

The reference for this section is Evans Chapter 5., and Sung-Jin Oh's 222A lecture notes, section 11.

1.1 Introduction to Sobolev Spaces

We begin with an introduction to the basics of Sobolev Spaces.

Definition 1. Let U be an open subset of $\mathbb{R}^d$, and $u \in \mathcal{D}'(U)$. The k^{th} order L^p-based Sobolev Norm of u is defined as:

$$||u||_{W^{k,p}U} = \left[\sum_{\alpha:|\alpha|<k} ||D^\alpha u||^p_{L^p(U)} \right]^{1/p}$$

Here, D^α is the weak derivative, and $D^\alpha u \in L^p(U)$.[1]

Remark. The sum over $||D^\alpha u||_{L^p}$ is motivated by its appearance in energy-method solutions to PDE's found in 222A.

Definition 2. The L^p-Sobolev space of order k on U is defined as

$$W^{k,p}(U) = \{u \in \mathcal{D}'(U) : ||u||_{W^{k,p}(U)} < \infty\}$$

Similarly, we call the subspace of $W^{k,p}(U)$ which vanish to appropriate order on the boundary

$$W_0^{k,p}(U) = \overline{C_c^\infty(U)}^{||\cdot||_{W^{k,p}(U)}} \subset W^{k,p}(U)$$

When $p = 2$, we have many extra analytical tools, since the Fourier transform is an L^2 isometry. This justifies special notation for this case.

Notation. We denote by $H^k(U) = W^{k,2}(U)$, and $H_0^k(U) = W_0^{k,2}(U)$.

We also have a special notation for inequalities that hold up to a multiplicative constant.

Notation. If, for some $c > 0$, $A \leq cB$, then $A \lesssim B$. If $A \lesssim B$ and $B \lesssim A$, then $A \simeq B$.

Prop 1.1. *Some basic facts about $W^{k,p}(U)$ and $H^k(U)$.*

i. For all $k \in \mathbb{Z}_{\geq 0}$ and $1 \leq p \leq \infty$, $(W^{k,p}(U), ||\cdot||_{W^{k,p}(U)})$ and $(W_0^{k,p}(U), ||\cdot||_{W^{k,p}})$ are Banach spaces.

ii. For all $k \in \mathbb{Z}_{\geq 0}$, and denoting $\langle u, v\rangle_{H^k(U)} = \sum_{\alpha:|\alpha|\leq k} \langle D^\alpha u, D^\alpha v\rangle_{L^2(U)}$, both $(H^k(U), \langle\cdot,\cdot\rangle_{H^k(U)})$ and $(H_0^k(U), \langle\cdot,\cdot\rangle_{H^k(U)})$ are Hilbert Spaces.

iii. (Fourier Analytic Characterization of H^k). If $u \in H^k(U)$, then $||u||_{H^k} \simeq ||\hat{u}||_{L^2} + |||\xi|^k\hat{u}||_{L^2} \simeq ||(1 + |\xi|^2)^{k/2}\hat{u}||_{L^2}$.

[1] This is equivalent to the other definition given in lecture: $||u||_{W^{k,p}(U)} = \sum_{\alpha:|\alpha|<k} ||D^\alpha u||_{L^p(U)}$.

. For (i), we first check that $||\cdot||_{W^{k,p}}$ is a norm. The triangle inequality may be verified by the elementary calculation, where $u, v \in W^{k,p}$:

$$
\begin{aligned}
||u+v||_{W^{k,p}} &= \left[\sum_{\alpha:|\alpha|\leq k} ||D^\alpha(u+v)||_{L^p}^p\right]^{1/p} \\
&\leq \left[\sum_{\alpha:|\alpha|\leq k} (||D^\alpha u||_{L^p} + ||D^\alpha v||_{L^p})^p\right]^{1/p} \\
&\leq \left[\sum_{\alpha:|\alpha|\leq k} ||D^\alpha u||_{L^p}^p\right]^{1/p} + \left[\sum_{\alpha:|\alpha|\leq k} ||D^\alpha v||_{L^p}^p\right]^{1/p} \\
&= ||u||_{W^{k,p}} + ||v||_{W^{k,p}}
\end{aligned}
$$

The first inequality follows from the fact that $||\cdot||_{L^p}$ is a norm. It is obvious that $||\lambda u||_{W^{k,p}} = |\lambda|||u||_{W^{k,p}}$.

It remains to check that $W^{k,p}$ is complete. Let $\{f_n\}_1^\infty$ be a Cauchy sequence in $W^{k,p}$. By definition, every $D^\alpha f_i \in L^p$, and since L^p is complete every $D^\alpha f_i$ converges to a function $f_\alpha \in L^p$. So, the claim is that when $\alpha = (0, \ldots, 0)$, we have convergence in L^p: $f_m \to f_{(0,\ldots,0)} := f \in W^{k,p}$. To see that $f \in W^{k,p}$, fix a test function $\phi \in C_0^\infty$, and integrate

$$
\begin{aligned}
\int f D^\alpha \phi dx &= \lim_{n\to\infty} \int f_n D^\alpha \phi dx \\
&= \lim_{n\to\infty} (-1)^{|\alpha|} \int (D^\alpha f_n)\phi dx \\
&= (-1)^{|\alpha|} \int f_\alpha \phi dx
\end{aligned}
$$

This shows that every Cauchy sequence of functions and all derivatives of index $|\alpha| < k$ converge in L^p, which proves convergence in $W^{k,p}$. For $W_0^{k,p}$, we need only check that f_α is compactly supported for all α. This is easily accomplished by replacing ϕ with $\varphi \in C^\infty$, and repeating the calculaton.

For (ii), we first check that $\langle\cdot,\cdot\rangle_{H^k}$ is an inner product. Letting $a, b \in \mathbb{C}$, and $f, g, h \in H^k$, we have

$$
\begin{aligned}
\langle af + bg, h\rangle_{H^k} &= \sum_{\alpha:|\alpha|\leq k} \langle D^\alpha(af+bg), D^\alpha h\rangle_{L^2} \\
&= \sum_{\alpha:|\alpha|\leq k} a\langle D^\alpha f, D^\alpha h\rangle_{L^2} + b\langle D^\alpha g, D^\alpha h\rangle_{L^2} \\
&= a\langle f, h\rangle_{H^k} + b\langle g, h\rangle_{H^k}
\end{aligned}
$$

That $\langle y, x\rangle_{H^k} = \overline{\langle x, y\rangle_{H^k}}$ follows from the same fact for the L^2 inner product, as does positivity. Completeness follows from (i), which shows that $(H^k, \langle\cdot,\cdot\rangle_{H^k})$ is a Hilbert space.

For (iii), we have to use some properties of the Fourier transform proved in 222A. Let $u \in H^k$. Since $u \in L^2$, we may write that $\widehat{(D^\alpha u)} = ((i\xi)^\alpha \hat{u})$. Clearly, we have that $||u||_{H^k} \geq C||\hat{u}||_{L^2} + |||\xi|^k\hat{u}||_{L^2}$, since the latter quantity contains only some of the terms present in the H^k norm. That $||\hat{u}||_{L^2} + |||\xi|^k\hat{u}||_{L^2} \geq C||(1+|\xi|^2)^{k/2}\hat{u}||_{L^2}$ follows from the Cauchy-Schwarz inequality and because $|\xi| > 0$, we have $(1+|\xi|^2)^{k/2} \leq C(1+|\xi|^2)^{k/2}$. From this, the chain of inequalities (choosing appropriate C so that all constant-dependent inequalities still hold) reads as

$$||u||_{H^k} \geq C\left(||\hat{u}||_{L^2}^2 + |||\xi|^k\hat{u}||_{L^2}^2\right)^{1/2} \geq C||(1+|\xi|^k)\hat{u}||_{L^2} \geq ||(1+|\xi|^2)^{k/2}\hat{u}||_{L^2}$$

All that remains is to show that $||(1+|\xi|^2)^{k/2}\hat{u}||_{L^2} \geq C||u||_{H^k}$. To see this, note that $(1+|\xi|^2)^{k/2} = \sum_{j=0}^{k/2} c_j|\xi|^{2j}$, pick the smallest $C = c_j$, and doing this once more for the sum that appears,

$$||(1+|\xi|^2)^{k/2}\hat{u}||_{L^2}^2 \geq C||\sum_{j=0}^{k}|\xi|^j\hat{u}||_{L^2}^2 \geq C' \sum_{\alpha:|\alpha|\leq k}||D^\alpha u||_{L^2}^2$$

Taking a square root completes the proof. □

Naturally, for a vector space like $W^{k,p}(U)$, we ask what the dual of this space is. By an appropriate definition, we can characterize the dual as being a Sobolev space of negative order.

Definition 3. For $k \in \mathbb{Z}_{\geq 0}$, $1 < p < \infty$, and U and open subset of $\mathbb{R}^d$, we define

$$||u||_{W^{-k,p}(U)} = \inf\left\{\sum_{\alpha:|\alpha|<k}||g_\alpha||_{L^p(U)} : u = \sum_{\alpha:|\alpha|<k} D^\alpha g_\alpha\right\}$$

and

$$W^{-k,p}(U) = \left\{u \in \mathcal{D}'(U) : u = \sum_{\alpha:|\alpha|<k} D^\alpha g_\alpha, g_\alpha \in L^p(U)\right\}$$

Remark. If $g \in L^p(U)$, then $D_{x^i}g \in W^{-1,p}(U)$. If $g \in W^{k,p}(U)$, then $D_{x^i}g \in W^{k-1,p}(U)$. In essence, we can characterize $W^{-k,p}$ as the space of L^p functions weakly differentiated up to k times.

With this in mind, we are able to prove the following proposition.

Prop 1.2. *For $k \in \mathbb{Z}_{\geq 0}$, $1 \leq p \leq \infty$, and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$,*

$$(W_0^{k,p}(U))^* \simeq W^{-k,p'}(U)$$

Proof. We show first that $(W_0^{k,p})^* \supseteq W^{-k,p'}(U)$. Let $v \in W^{-k,p'}(U)$, and $u \in W_0^{k,p}(U)$. By definition, we may write $v = \sum_{\alpha:|\alpha|<k} D^\alpha g_\alpha$. Testing v against u, we find that

$$\begin{aligned}\langle v, u\rangle = \int_U vudx = \sum_{\alpha:|\alpha|<k} \int_U D^\alpha g_\alpha udx &= \sum_{\alpha:|\alpha|<k} \int_U (-1)^\alpha g_\alpha D^\alpha udx \\ &\leq \sum_{\alpha:|\alpha|<k} ||g_\alpha||_{L^{p'}} ||D^\alpha u||_{L^p} \\ &\leq c||v||_{W^{-k,p'}} ||u||_{W^{k,p}}\end{aligned}$$

Here, the third equality follows by integrating by parts, using the fact that $\mathrm{spt}(u)$ is compact in U to disregard boundary terms. The first inequality is a direct application of Hölder's inequality (A.4). The second is the definition of the Sobolev norm, aggregating the constants from each term into c. Thus, we have shown that every v in $W^{-k,p'}(U)$ is a bounded linear functional on $W_0^{k,p}$, i.e. an element of the dual space.

To show that $(W_0^{k,p})^* \subset W^{-k,p'}$, we apply the Hahn-Banach theorem (A.2). First, define the bounded linear functional $\ell : W_0^{k,p} \to \mathbb{R}$, and let $u \in C_0^\infty(U)$, with the end goal that $\ell(u) = \langle v, u\rangle = \sum_{\alpha:|\alpha|<k} (-1)^\alpha \langle g_\alpha, D^\alpha u\rangle$. To that end, we define (where $K(k)$ is the total number of multi-indices up to order k):

$$\begin{aligned} T: \quad & C_0^\infty(U) \to L^p(U)^{K(k)} \\ & u \mapsto (u, D_{x^1}u, \ldots, D_{x^d}u, \ldots D^\alpha u)\end{aligned}$$

We have that $||T(u)|| \leq c||u||_{W^{k,p}}$. Furthermore, T is injective, and an isomorphism onto its image, i.e., $(C_0^\infty(U), ||\cdot||_{W^{k,p}}) \sim (T(C_0^\infty(U)), ||\cdot||)$. So, we may send ℓ to $\tilde{\ell} : T(C_0^\infty(U)) \to \mathbb{R}$ by composing with this isomorphism. In particular, $\tilde{\ell}(Tu) = \ell(u)$ tells us that $\tilde{\ell}$ is similarly bounded. Now, by the Hahn-Banach theorem, $\tilde{\ell}$ extends to the bounded linear functional $\tilde{\tilde{\ell}} : (L^p(U))^{\otimes K} \to \mathbb{R}$. By definition, $\tilde{\tilde{\ell}} \in \left((L^p(U))^{\oplus K}\right)^* = \{\tilde{v} = \sum_{\alpha:|\alpha|<k} \tilde{g}_\alpha : \tilde{g}_\alpha \in L^{p'}(U)\}$. So, for some $\tilde{u} \in (L^p(U))^{\oplus K}$, the pairing with $\tilde{v}$ is exactly $\langle \tilde{v}, \tilde{u}\rangle = \sum_{\alpha:|\alpha|<k} \langle g_\alpha, u_\alpha\rangle$, where $\tilde{u}_\alpha = D^\alpha u$. So,

$$\ell(u) = \tilde{\ell}(Tu) = \tilde{\tilde{\ell}}(Tu) = \sum_{\alpha:|\alpha|<k} \langle \tilde{g}_\alpha, (Tu)\alpha\rangle = \sum_{\alpha:|\alpha|<k} \langle \tilde{g}_\alpha, D^\alpha u\rangle$$

If we choose $g_\alpha = (-1)^{|\alpha|}\tilde{g}_\alpha$, we have shown the remainder of the proof. □

1.2 Existence and Uniqueness Problems

The concrete objective of this section is to explore the duality relationship between the existence and uniqueness of solutions to linear equations on Banach spaces. In particular, apriori estimates of the dual problem prove the existence of solutions to the direct problem, and vice-versa (under certain conditions).

Prop 1.3. *Let X, Y be Banach Spaces, and let $P : X \to Y$ be a bounded linear operator. Likewise, let $P^* : Y^* \to X^*$ be the adjoint of P. Suppose that there exists $c > 0$ such that $||u||_X \leq c||Pu||_Y$ for all $u \in X$. Then the following hold:*

i. *(Uniqueness for $Pu = f$) If $u \in X$, $Pu = 0 \Rightarrow u = 0$.*

ii. *(Existence for $P^*v = g$) For all $g \in X^*$, there exists $v \in Y^*$ such that $P^*v = g$, and $||v||_{Y^*} \leq c||g||_{X^*}$.*

Proof. The proof of (i) is clear, since $||u||_X \leq 0$, $u = 0$ since X is normed.

As for (ii), we again apply the Hahn-Banach theorem. In particular, our objective is to find $v \in Y^*$ such that for all $u \in X$: $P^*v = g \Leftrightarrow \langle P^*v, u\rangle = \langle g, u\rangle = \langle v, Pu\rangle$. To that end, define $\ell : P(X) \to \mathbb{R}$, where $\ell(Pu) = \langle g, u\rangle$. Since P is injective by (i), ℓ is well-defined. By definition, if $||Pu||_Y \leq 1$, we have that

$$|\ell(Pu)| = |\langle g, u\rangle| \leq ||g||_{X^*}||u||_X \leq c||g||_{X^*}||Pu||_Y \leq c||g||_{X^*}$$

So, by the Hahn-Banach theorem, there exists $v \in Y^*$ such that $\langle v, Pu\rangle = l(Pu) = \langle g, u\rangle$ for all $u \in X$, and $||v||_{Y^*} \leq c||g||_{X^*}$. □

Definition 4. Let X be a normed vector space with member x, and let $\hat{x} : X^* \to \mathbb{C}$ denote $\hat{x}(f) = f(x)$. Let $\hat{X} = \{\hat{x} : x \in X\}$. X is called **reflexive** if $\hat{X} = X^{**}$.

If we want existence for the direct problem, we take the easy way, and assume X is reflexive, which yields Proposition 1.4. In general, $\hat{X} \subseteq X^{**}$.

Prop 1.4. *Let X, Y be Banach Spaces, with X reflexive, and Let $P : X \to Y$ be a bounded linear operator. Likewise, let $P^* : X^* \to Y^*$ be the adjoint of P. Suppose that there exists $c > 0$ such tht $||v||_{Y^*} \leq c||P^*v||_{X^*}$. Then the following hold:*

i. *(Uniqueness for $P^*v = g$) If $v \in Y^*$, $P^*v = 0 \Rightarrow v = 0$.*

ii. *(Existence for $Pu = f$) For all $f \in Y$, there exists $u \in X$ such that $Pu = f$, and $||u||_X \leq c||f||_Y$.*

. Exercise. □

Remark. All Sobolev Spaces $W_0^{k,p}$ for $1 < p < \infty$ are reflexive. This will be a homework problem.

Notation. Let $P : X \to Y$ be a linear operator, and P^* its associated adjoint. With $U \subset Y$, and $V \subset X^*$, we define the following sets:

$$U^\perp = \{v \in Y^* : \langle v, f\rangle = 0 \,\forall f \in U\}$$
$$^\perp V = \{u \in X : \langle g, u\rangle = 0 \,\forall g \in V\}$$

Remark. $\text{range}(P)^\perp = \ker(P^*)$, and $\ker(P) = ^\perp \text{range}(P^*)$. As a consequence of this fact, if $\ker P^* = \{0\}$, then $\text{range}(P)^\perp = \{0\} \Leftrightarrow \overline{\text{range}(P)} = Y$.

It's worth noting that in finite dimensions, $\overline{\text{range}(P)} = \text{range}\, P$, which provides the simpler duality relation between uniqueness and existence of solutions for linear operators. In infinite dimensions, this does not always hold, which is what our boundedness estimate provides. There is no loss of generality for deriving existence for P from the qualitative bound

$$||v||_{Y^*} \leq c||P^*v||_{X^*} \tag{1}$$

Notation. We denote by $B_X = \{x \in X : ||x||_X < 1\}$.

Prop 1.5. *Let X, Y be Banach Spaces, and $P : X \to Y$ a bounded linear operator. If $P(X) = Y$, then there exists $c > 0$ such that (1) holds.*

Proof. $||P^*v||_{X^*} = \sup_{\overline{B_X}} |\langle P^*v, u\rangle| = \sup_{\overline{B_X}} |\langle v, Pu\rangle|$. T is a surjective linear map between Banach spaces, and is therefore open by the Open mapping theorem (A.3). Thus, $P(B_X)$ is open and contains 0. Thus, there exists $c > 0$ such that $P(B_X) \supseteq cB_Y$, which implies

$$||P^*v||_{X^*} = \sup_{\overline{B_X}} |\langle P^*v, u\rangle| = \sup_{\overline{B}_X} |\langle v, Pu\rangle| \geq \sup_{f \in cB_Y} |\langle v, f\rangle| = c||v||_{Y^*}$$

which completes the proof. □

Example. We now examine the solvability of the equation $-u'' = f$ in $H_0^1((0,1))$. Note that $||u||_{H^1}^2 = ||u||_{L^2}^2 + ||u'||_{L^2}^2$, and that $(H_0^1)^* = H^{-1}$. Using $X = H_0^1, Y = H^{-1}$, we consider $P = -\partial_x^2$, and claim that if $-u'' = f$ for $u \in H_0^1$, then

$$||u||_{H^1} \leq c||f||_{H^{-1}}.$$

The proof is an application of the energy method. A simple integration by parts yields that

$$\int -u''udx = \int fudx = \int (u')^2dx = ||u'||_{L^2}^2$$

To obtain the previous inequality from what we have just derived, we use the fact that u is zero on the boundary, and so $u(x) = \int_0^x u'(y)dy$. Using the Cauchy-Schwarz inequality,

$$|u(x)| \leq \int_0^1 |u'(y)|dy \leq ||u'||_{L^2}$$

which implies that

$$\int_0^1 |u|^2dy \leq \sup_{[0,1)} |u|^2 \leq ||u'||_{L^2}^2,$$

in turn implying that

$$||u||_{H^1}^2 \leq c|\langle f, u\rangle| \leq c||f||_{H^{-1}} + ||u||_{H^1}$$

which completes the proof after dividing out a term $||u||_{H^1}$.

From this, we can deduce from the inequality above, and Proposition 1.3 that $-u'' = 0$ and $u \in H_0^1 \Rightarrow u = 0$. From the inequality and Proposition 1.4, we should compute P^*, and obtain existence for the dual problem. Explicitly, we ue the fact that H_0^1 is reflexive, and compute for $u \in H_0^1$, that

$$\langle v, Pu\rangle = \int_0^1 v(-u'')dx = \int_0^1 v'u'dx = \int_0^1 -v''udx = \langle P^*v, u\rangle$$

So, $P^* = -\partial_x^2$ as well, meaning the problem is entirely self-adjoint, and $Y^* = H_0$ 1. This gives that $\forall f \in H^{-1}$, there exists $u \in H_0^1$ such that $Pu = f$, by applying Proposition 1.4.

This hints at the Poincaré inequality, which we will explore shortly.

1.3 Approximation Theorems

1.3.1 Convolution and Mollifiers

Definition 5. Let $\varphi \in C_0^\infty(U)$, with $\int_U \varphi = 1$. We define the family of functions $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi\left(\frac{x}{\varepsilon}\right)$. Note that $\int_U \varphi_\varepsilon dx = 1$ for every ε.

Lemma 1.6. *Let $\varphi \in C_0^\infty$, with $\int \varphi dx = 1$, $u \in L^p(\mathbb{R}^d$ where $1 \leq p \leq \infty$, and φ_ε as in Definition 5. As $\varepsilon \to 0$, $||\varphi_\varepsilon \star u - u||_{L^p} \to 0$.*

Before proving the lemma, we note that translations are continuous in L^p.

Lemma 1.7. $\lim_{|z|\to 0} ||u(x-z) - u(x)||_{L^p} = 0$ *for $u \in L^p$.*

. Clearly, the conditions of the dominated convergence theorem are satisfied, choosing u_n to be a sequence of functions $u_n(x) = u(x - z_n)$, where $z_n \to 0$. It suffices to dominate u_n by $v(x) = |u(x)| \cdot \mathbf{1}_{B_1(x)}$, under the assumption that z_n is a sequence which is of distance at most 1 from x. □

Proof. (Lemma 1.6) Consider:

$$\begin{aligned}\varphi_\varepsilon \star u - u &= \int_U u(x-y)\varphi_\varepsilon(y)dy - u(x) \\ &= \int_U (u(x-y) - u(x)\varphi_\varepsilon(y)dy\end{aligned}$$

Therefore,

$$\begin{aligned}||\varphi_\varepsilon \star u - u||_{L^p} &= \left\| \int_U (u(x-y) - u(x)\varphi_\varepsilon(y)dy \right\|_{L^p} \\ &\leq \int_U ||u(\cdot - y) - u(\cdot)||_{L^p}|\varphi_\varepsilon(y)|dy\end{aligned}$$

The inequality above is a direct application of the Minkowski inequality. First, we note that $\text{spt}(\varphi_\varepsilon) \to \{0\}$ as $\varepsilon \to 0$. Furthermore, by the L^p-continuity of translations, the entire integrand converges to zero. □

Definition 6. A **partition of unity** on an open set $U \subset \mathbb{R}^d$ is a family of functions $\{\chi_\alpha\}_{\alpha \in A}$ such that the following hold:

1. $\sum_{\alpha \in A} \chi_\alpha(x) = 1$ for all $x \in U$.
2. For every $x \in U$, only finitely many χ_α are nonzero at x.

If $\{U_\alpha\}_{\alpha \in A}$ is an open cover of U, $\{\chi_\alpha\}_{\alpha \in A}$ is called **subordinate to** $\{U_\alpha\}_{\alpha \in A}$ if $\text{spt}\, \chi_\alpha \subseteq U_\alpha$ for all α. If $\chi_\alpha \in C_0^\infty$, then $\{\chi_\alpha\}_{\alpha \in A}$ is called a **smooth partition of unity**.

Lemma 1.8. *Let $\{U_\alpha\}_{\alpha\in A}$ be an open covering of $U \subset \mathbb{R}^d$. Then there exists a smooth partition of unity subordinate to $\{U_\alpha\}_{\alpha\in A}$.*

Proof. Largely omitted. To see this, start from a continuous subordinate partition of unity, and mollify it until you get a smooth partition. ☐

1.3.2 Density Theorems

In what follows, we prove four density theorems, and an extension theorem. The goal is to provide tools for representing members $u \in W^{k,p}(U)$ by objects with prescribed smoothness and support properties.

Theorem 1.9. *Let $k \in \mathbb{Z}_{\geq 0}$, $1 \leq p < \infty$. Then*

i. $C^\infty(\mathbb{R}^d)$ *is dense in* $W^{k,p}(\mathbb{R}^d)$.

ii. $C_0^\infty(\mathbb{R}^d)$ *is dense in* $W^{k,p}(\mathbb{R}^d)$.

Proof. (i) is a rote application of mollifiers. (ii) will be homework. The main step is to approximate f by $f\chi(1/R)$, with $\chi \in C_c^\infty$, and $\chi(0) = 1$. ☐

Theorem 1.10. *Let $k \in \mathbb{Z}_{\geq 0}$, $1 \leq p < \infty$, and U be an open set in $\mathbb{R}^d$. Then $C^\infty(U)$ is dense in $W^{k,p}(U)$.*

Proof. Let $u \in W^{k,p}(U)$, and fix $\epsilon > 0$. We want to find $v \in C^\infty(U)$ such that $||u-v||_{W^{k,p}(U)} \leq c\epsilon$. To that end, consider the family of opens $U_j = \{x \in U : \operatorname{dist}(x, \partial U) < \frac{1}{j}\}$, and define $V_j = U_j \setminus \overline{U_{j+2}}$. Then, since $U \subseteq \bigcup_{j=1}^\infty V_j$, we may choose χ_j to be a partition of unity of U subordinate to V_j, and write

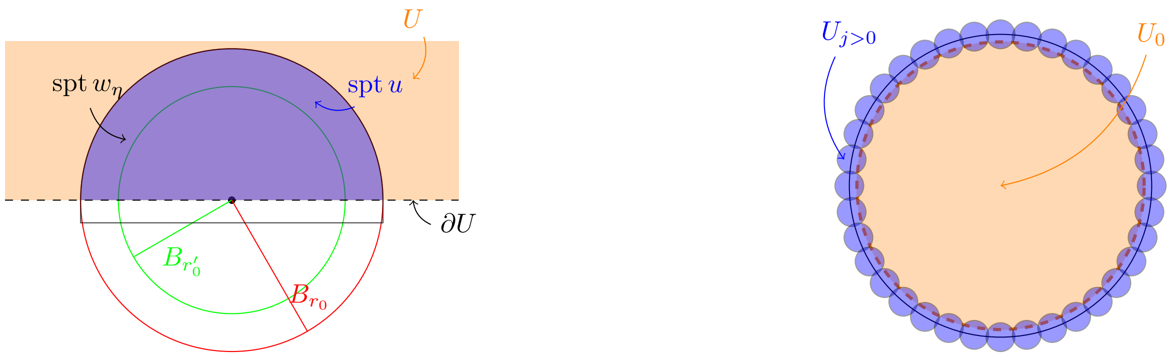
$$u = \sum_{j=1}^\infty u\chi_j := \sum_{j=1}^\infty u_j$$

Note that because $\operatorname{spt}\chi_j \subseteq V_j$, $\operatorname{spt} u_j \subseteq V_j$, Furthermore, $u_j \in C_0^\infty(\mathbb{R}^d)$, since it is smoothly extended by 0 outside V_j.

Now, we define a mollifier $\varphi \in C_0^\infty(\mathbb{R}^d)$, with the usual $\int \varphi dx = 1$, and $\operatorname{spt}\varphi \subseteq B_1(0)$. This automatically gives us that $\operatorname{spt}\varphi_{\varepsilon_j} \subseteq B_{\epsilon_j}(0)$, for prescribed ε_j. We prescribe ε_j by defining a new $v_j = \varphi_{\varepsilon_j} * u_j$, and choosing each ε_j so that $||u_j - v_j||_{W^{k,p}(U)} \leq 2^{-j}\epsilon$, and $\operatorname{spt} v_j \subseteq \tilde{V}_j = U_{j-1} \setminus \overline{U_{j+2}}$. With these prescribed, we take $v = \sum_{j=1}^\infty v_j$. First, v is well-defined since $\tilde{V}_j$ is locally finite. Second, we compute

$$||v-u||_{W^{k,p}(U)} \leq \sum_{j=1}^\infty ||v_j - u_j||_{W^{k,p}(U)} \leq 2^{-j}\epsilon = c\epsilon$$

So, v has the desired convergence and smoothness properties, so we are done. ☐



(a) Illustration of Step 2.

(b) Open Cover of U

Figure 2: Illustration of the proof of Theorem 1.11

One issue with Theorem 1.10 is its lack of control over v near the boundary of U. We attempt to resolve this in the following theorem, but first, we define some terms.

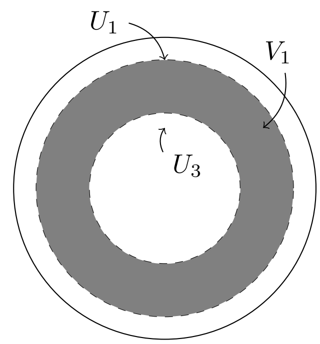


Figure 1: U_j and V_j in the proof of Theorem 1.10

Notation. $C^\infty(\overline{U}) = \{u : U \to \mathbb{R} : u$ is the restriction of a function $\tilde{u} \in C^\infty(\tilde{U}), \tilde{U} \supseteq \overline{U}\}$.

Definition 7. We say that ∂U is of class C^k if it is locally the graph of a C^k function.

Theorem 1.11. *Let $k \in \mathbb{Z}_{\geq 0}$, $1 \leq p \leq \infty$, and U a bounded open subset in $\mathbb{R}^d$, with ∂U of class C^1.*[2] *Then $C^\infty(\overline{U})$ is dense in $W^{k,p}(U)$.*

Proof. The proof proceeds in two steps. The first is to reduce the problem for a general U to a region where we may consider U to be the graph of a C^1 function. The second is to apply our approximation theorems to these simpler regions, before stitching the function back together.

By the definition of C^1-regularity, and the fact that U is bounded, we may cover ∂U by a finite family of open balls $\{B_{r_j}(x_j)\}_{j=1}^J$, in each of which U may by represented as the region above a C^1 graph. Calling $U_j = B_{r_j}(x_j)$, we choose an open set U_0 such that $U \supseteq U_0 \supseteq U \setminus \bigcup_{j=1}^J U_j$. Then $\{U_j\}_{j=0}^J$ is an open cover of U (which is illustrated in Figure 2b), so we may take $\{\chi_j\}_{j=0}^J$ to be a partition of unity subordinate to U_j, and–as in the proof of Theorem 1.10–write:

$$u = \sum_{j=0}^{J} u\chi_j := u_0 + \sum_{j=1}^{J} u_j$$

u_0 already has compact support, so we are free to apply the previous results. For $u_{j>0}$, we need to give a more explicit description of the boundary.

[2]This can be relaxed to give U a Lipschitz boundary.

For this portion, we fix $u = u_j$, and make a change of coordinates so that we are centered at the origin, with r_0 and r_0' defined appropriately as in Figure 2a. We note that $\partial U = \{x^d = \Gamma(x^1, \dots x^{d-1}\}$ possibly after a change of coordinates, where Γ is the C^k graph function. Letting e^d represent the unit coordinate vector in the direction of x^d, we make an approximation in two parts. First, define $w_\eta(x) = u(x + \eta e^d)$. Note that as $\eta \to 0$, by Lemma 1.7, $||u - w_\eta||_{W^{k,p}(U \cap B_{r_0})} < \frac{1}{2}\epsilon$ Moreover, w_η is defined on the set $B_{r_0'} \cap U - \eta e^d$. Second, we choose $v = \varphi_{r_0'} * w_\eta$, where φ is a mollifier. Then, for $r_0' \ll \eta$, v is well-defined on $B_{r_0'} \cap \{x^d > \Gamma(x^1, \dots, x^{d-1})\}$, and $||v - w_\eta||_{W^{k,p}(U \cap B_{r_0})} < \frac{1}{2}\epsilon$. So, an application of the triangle rule gives that

$$||u - v||_{W^{k,p}(U)} \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon \leq \epsilon$$

And, since $v \in C^\infty(\overline{V \cap \{x^d > \Gamma(x^1, \dots, x^d)\}})$, we are done. □

1.3.3 Trace and Extension Theorems

Extension theorems can be roughly thought of as tools allowing us to handle $u \in W^{k,p}(U)$ when U is a bounded domain.

Theorem 1.12. *Let $k \in \mathbb{Z}_{\geq 0}$, $1 \leq p < \infty$, U a bounded domain in $\mathbb{R}^d$ with C^k boundary, and V be an open set containing $\overline{U}$. Then there exists $E : W^{k,p}(U) \to W^{k,p}(\mathbb{R}^d)$ such that the following hold:*

i. (Extension property) $Eu_{|U} = u$.

ii. (Linearity and Boundedness) E is linear, and $||Eu||_{W^{k,p}(\mathbb{R}^d)} \leq c||u||_{W^{k,p}(U)}$.

iii. (Support) $\operatorname{spt}(Eu) \subseteq V$.

Proof. Observe that by Theorem 1.11, and the fact that U is bounded, it suffices to consider $u \in C^\infty(\overline{U})$. The proof proceeds in two steps. First, we reduce to the half-ball case, and second, we prove extension for the half-ball case.

To reduce our problem to the case of the half-ball, it suffices to construct an open cover $\{U_0, \dots, U_J\}$ as in the proof of Theorem 1.11, with similarly constructed partition of unity $\{\chi_j\}_{j=0}^J$, and $u_j := u\chi_j$. Notably, $u_0 \in W^{k,p}(\mathbb{R}^d)$ since it is smoothly extended by 0, and $u_k \in C^\infty(\overline{U})$, and $\operatorname{spt} u_k \subseteq U_k \cap U$. After making a change of coordinates to $y^j = x^j$ for $1 \leq j < d$, and $y^d = x^d - \Gamma(x^1, \dots, x^{d-1})$, we see that $U_k \cap U \mapsto \{y \in B_{\tilde{r}}(0) : y^d > 0\} := \tilde{U}_k$, and $x \mapsto y$ is C^k, with smooth U_j. Therefore, applying the chain rule, we find that $u_j(y) = u_j(x(y))$ satisfies

$$||u_j(y)||_{W_y^{k,p}(\tilde{U}_j)} \leq c||u_k(x)||_{W_x^{k,p}(U_j \cap U)}$$

Thus, it suffices to consider the half-ball case.

The second step is to actually extend u in the case of the half-ball. Here, we define $U = B_r^+(0)$, and $W = B_{r/2}^+(0)$, such that $\operatorname{spt} u \subset W$. In order to extend u, we use the higher order reflection method, defining:

$$Eu = \tilde{u} = \begin{cases} u(x) & x^d > 0 \\ \sum_{j=0}^{k} \alpha_j u(x^1, \dots, x^{d-1}, -\beta_j x^d) & x^d < 0 \end{cases}$$

Our objective here is to match up the normal derivatives of $\tilde{u}$ with u up to order k, i.e. set $u(x^1, \dots, x^{d-1}, 0+) = \sum_{j=0}^{k} \alpha_j u(x^1, \dots, x^{d-1}, 0-)$, and likewise for all derivatives $\partial^j_{x^d} u = (-\beta_j)^j \partial_{x^d} u$. This sets up the following matrix equation for our coefficients α and β.

$$\begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ -\beta_1 & \cdots & -\beta_1 \\ \vdots & \vdots & \vdots \\ (-\beta_k)^k & \cdots & (-\beta_k)^k \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \vdots \\ \vdots \\ \alpha_k \end{pmatrix}$$

This is the Vandermonde matrix, and if all β_j's are distinct, the matrix is invertible, which implies that the existence of $\alpha_1, \dots, \alpha_k$ such that the equation holds. The existence of such coefficients defines $\tilde{u}$ on $B_r(0)$, extending u and matching up derivatives to order C^k. Finally, to ensure extension to all of $\mathbb{R}^d$, we apply a cutoff function (a lá Urysohn's lemma) χ_V, such that $\chi_V = 1$ on U, and $\operatorname{spt} \chi_V \subset V$. □

Now, we move on to discussing trace theorems, which essentially revolve around the restriction of functions $u \in W^{1,p}(U)$ to ∂U. This is interesting in part because the measure of $\mu(\partial U) = 0$, so using only L^p theory to deal with differentiability on the boundary gives little help, since L^p equivalence is almost everywhere.

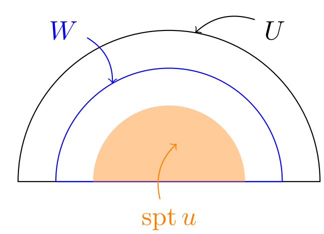


Figure 3: Illustration of the proof of Theorem 1.12.

Definition 8. Let $u \in C^1(\overline{U})$, and let U be a domain with C^1 boundary. Then the **trace of u on the boundary of U** is defined $\operatorname{tr}_{\partial U}(u) = u_{|\partial U}$.

Our objective is to first extend this definition to all of $W^{1,p}(U)$. We note that $\operatorname{tr}_{\partial U}$ is clearly linear, and will often write tr when ∂U is clear from context. Furthermore, whenever the L^p norm is used on a manifold of dimension less than the ambient space, it is assumed that integration is with respect to the volume fold of the manifold.

Theorem 1.13 (Nonsharp Trace Theorem)**.** *Let U be a bounded, open subset of $\mathbb{R}^d$, with ∂U of class C^1, and $1 < p < \infty$. Then for $u \in C^1(\overline{U})$,*

$$||\operatorname{tr}_{\partial U} u||_{L^p(\partial U)} \lesssim ||u||_{W^{1,p}(U)}$$

As a consequence of this inequality, the following facts hold:

i. *$\operatorname{tr}_{\partial U}$ is extended uniquely by continuity and density of $C^1(\overline{U}) \subseteq W^{1,p}(U)$ to $\operatorname{tr}_{\partial U} : W^{1,p}(U) \to L^p(\partial U)$*

ii. $u \in W_0^{1,p} \Leftrightarrow \operatorname{tr}_{\partial U} u = 0.$

Proof. Evans section 5.5. □

Note that this extension is *not* surjective. $\operatorname{img}(\operatorname{tr}) \subsetneq L^p(\partial U)$.

We direct our attention to a sharp version of Theorem 1.13 in the setting where $p = 2$. This opens up the world of Fourier analysis, and eventually leads to the world of fractional-order Sobolev Spaces. We prove a Sharp Trace theorem for the half-space $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^d > 0\}$, and denote $\partial U = \{(x', 0) \in \mathbb{R}^d\} \simeq \mathbb{R}^{d-1}$.

Notation (Fourier Transform)**.** The convention used for the Fourier and Inverse Fourier transforms is as follows: $\hat{u} = \int u(x) e^{-ix\xi} dx$, and $u(x) = \int \hat{u} e^{i\xi x} \frac{d\xi}{2\pi}$.

Theorem 1.14 (Sharp Trace Theorem)**.** *When* $u \in C^1(\overline{\mathbb{R}^d_+}) \cap H^1(\mathbb{R}^d_+)$*, we have:*

$$||\operatorname{tr} u||_{H^{1/2}(\mathbb{R}^{d-1})} \lesssim ||u||_{H^1(\mathbb{R}^d_+)}$$

Proof. Let u be as in the Theorem statement. Using Theorem 1.12, we may extend u to $\tilde{u} \in C^1(\mathbb{R}^d)$ such that $||\tilde{u}||_{H^1(\mathbb{R}^d)} \lesssim ||u||_{\mathbb{R}^d_+}$. Then, we may write

$$\operatorname{tr} u = u(x', 0) = \tilde{u}(x', 0) = \int [\mathcal{F}_{x^d} \tilde{u}](x', \xi^d) \frac{d\xi^d}{2\pi}$$

Furthermore,

$$[\mathcal{F}_{x'} \operatorname{tr} u](\xi') = \int [\mathcal{F} \tilde{u}](\xi', \xi^d) \frac{d\xi^d}{2\pi}$$

Using the Fourier characterization from Proposition 1.1, we may write

$$
\begin{aligned}
||\operatorname{tr} u||_{H^s} &\simeq ||(1+|\xi'|^2)^{s/2}[\mathcal{F}_{x'} \operatorname{tr} u](\xi')||_{L^2_{\xi'}} \\
&= \left\| (1+|\xi'|^2)^{s/2} \int [\mathcal{F}\tilde{u}](\xi', \xi^d) \frac{d\xi^d}{2\pi} \right\|_{L^2_{\xi'}} \\
&\simeq \left\| \int [\mathcal{F}\tilde{u}](\xi', \xi^d)(1+|\xi'|^2)^{s/2} d\xi^d \right\|_{L^2_{\xi'}} \\
&= \left\| \left\| [\mathcal{F}\tilde{u}](\xi', \xi^d)(1+|\xi'|^d)^{s/2} \frac{(1+|\xi'|^2+|\xi^d|^2)^{1/2}}{(1+|\xi'|^2+|\xi^d|^2)^{1/2}} \right\|_{L^1_{\xi^d}} \right\|_{L^2_{\xi'}} \\
&\leq \left\| \left\| \frac{(1+|\xi'|^2)^{s/2}}{(1+|\xi|^2)^{1/2}} \right\|_{L^2_{\xi^d}} \left\| (1+|\xi|^2)^{1/2}[\mathcal{F}\tilde{u}] \right\|_{L^2_{\xi^d}} \right\| \\
&= \left\| \left(\int \frac{(1+|\xi'|^2)^s}{1+|\xi'|^2+|\xi^d|^2} d\xi^d \right)^{1/2} ||(1+|\xi|^2)^{1/2}[\mathcal{F}\tilde{u}]||_{L^2_{\xi^d}} \right\|_{L^2_{\xi'}} \\
&\leq \left(\sup_{\xi' \in \mathbb{R}^{d-1}, s \in \mathbb{R}} \left[\int \frac{(1+|\xi'|^2)^s}{1+|\xi'|^2+|\xi^d|^2} d\xi^d \right] \right) ||u||_{H^1(\mathbb{R}^d_+)} \\
&\simeq ||u||_{H^1(\mathbb{R}^d_+)}
\end{aligned}
$$

□

Theorem 1.15 (Extension from the Boundary)**.** *There exists a bounded linear map* $\operatorname{ext}_{\partial U} : H^{1/2}(\mathbb{R}^{d-1}) \to H^1(\mathbb{R}^d_+)$ *such that* $\operatorname{tr}_{\partial U} \circ \operatorname{ext}_{\partial U} = \operatorname{id}$.

Proof. Here, we use the Poisson Semigroup. In particular, define $g \in \mathcal{S}(\mathbb{R}^{d-1})$, and $u = \operatorname{ext}_{\partial U}(g)$, with $[\mathcal{F}_{x'} u](\xi', x^d) = \eta(x^d) e^{-x^d|\xi'|}\hat{g}(\xi')$. Here, η is a smooth cutoff function with $\eta(|s| < 1) = 1$, and $\eta(|s| > 2) = 0$. Our objective is that show that $u \in H^1(\mathbb{R}^d_+)$ if and only if the following statements hold:

i. $u, \partial_{x^1} u, \ldots, \partial_{x^{d-1}} \in L^2$.
ii. $\partial_{x^d} u \in L^2$.

For (i), assume that $u \in H^1(\mathbb{R}^d_+)$. Then

$$
\begin{aligned}
||u||^2_{L^2} + ||\partial_{x^1} u||^2_{L^2} + \ldots + ||\partial_{x^{d-1}} u||^2_{L^2} &\simeq ||(1+|\xi'|)^2)^{1/2}[\mathcal{F}_{x'} u](\xi', x^d)||^2_{L^2_{\xi'} L^2_{x^d}} \\
&= ||(1+|\xi'|^2)^{1/2} \eta(x^d) e^{-x^d|\xi'|}\hat{g}(\xi')||^2_{L^2_{\xi'}, L^2_{x^d}} \\
&= \left\| (1+|\xi'|^2)^{1/4} ||\eta(x^d) e^{-x^d|\xi'|}||_{L^2_{x^d}} (1+|\xi'|^2)^{1/4}\hat{g}(\xi') \right\|^2_{L^2_{\xi'}}
\end{aligned}
$$

We want to put a uniform bound on $(1+|\xi'|^2)^{1/4}||\eta(x^d)e^{-x^d|\xi'|}||_{L^2_{x^d}}$ for every ξ'. By the compact support of η, we have the trival inequality:

$$||\eta(x^d)e^{-x^d}||^2_{L^2_{x^d}} \lesssim 1$$

Furthermore, writing out the L^2-norm, and making a substitution of variables inside the integral, we arrive at the substitution inequality:

$$\int (\eta(x^d))^2 e^{-2x^d|\xi'|} dx^d \lesssim \frac{1}{|\xi'|}$$

From these two inequalities, we deduce that

$$||\eta(x^d)e^{-x^d|\xi'|}||_{L^2_{x^d}} \lesssim \min\{1, |\xi'|^{-1/2}\} \lesssim (1+|\xi'|)^{-1/2}$$

So, terms in our initial equality cancel as follows:

$$\left\| \cancel{(1+|\xi'|^2)^{1/4}} \cancel{||\eta(x^d)e^{-x^d|\xi'|}||_{L^2_{x^d}}} \cancel{(1+|\xi'|^2)^{1/4}} \hat{g}(\xi') \right\|^2_{L^2_{\xi'}} \simeq ||\hat{g}(\xi')||_{L^2_{\xi'}}$$

This proves (i), since we may unravel the chain of definitions in the reverse direction exactly the same way.

To see (ii), write $\partial_{x^d} u = \partial_{x^d}(\eta(x^d)v) = \eta'(x^d)v + \eta v'$, $\mathcal{F}_{x'}v = e^{-x^d|\xi'|}\hat{g}(\xi)$. Each term of u' may be bounded, with $\eta'(x^d)v \leq ||v||_{L^2(x^d \in \operatorname{spt}\eta)}$, and

$$||\eta\partial_{x^d}v||_{L^2_{x'}L^2_{x^d}} = ||\eta\partial_{x^d}(e^{-x^d|\xi'|}\hat{g}(\xi'))||_{L^2_{x^d}L^2_{\xi'}} = ||\eta(x^d)|\xi'|e^{-x^d|\xi'|}\hat{g}(\xi')|| \lesssim c||g||_{H^{1/2}}$$

The final inequality here follows from (i). □

To generalize the above theorems to L^2-based Sobolev spaces, we need fractional Sobolev spaces on a C^1 boundary, under C^1 straightening diffeomorphism to the half-space. The indepedence of the norm under this diffeomorphism follows from some concepts in interpolation theory, which is in a book of Stein from 1970.

For $p \neq 2$, $\operatorname{img}(\operatorname{tr}_{\partial U} W^{1,p}(U)) = B_p^{1-1/p,p}(\partial U)$, the L^p-Besov space with regularity index of order $1 - 1/p$, and summability index p. This is also in Stein.

1.4 Sobolev Inequalities

1.4.1 $1 \leq p < d$

In a nutshell, Sobolev inequalities are quantitative generalizations of the Fundamental theorem of Calculus, allowing us to control the size of a function by the growth of it's derivative.

Theorem 1.16 (Gagliardo-Nirenberg-Sobolev Inequality). *For $d \geq 2$, $u \in C_0^\infty(\mathbb{R}^d)$, we have that for* *a constant c_d which depends only on the dimension d:*

$$||u||_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq c_d ||Du||_{L^1(\mathbb{R}^d)}$$

Remark. The factor of $\frac{d}{d-1}$ can be derived using *dimensional analysis*. It's basically a statement about the fact derivatives have dimensions $[D] \sim \frac{1}{\lambda}$, where λ is the scalaing factor, and that the L^p norm has dimension $[|| \cdot ||_{L^p}] \sim [\lambda]^{d/p}$. So $[||D \cdot ||_{L^p}] \sim [\lambda]^{d-1}$, and the rest follows from equating the dimensions.[3]

The key ingredient in the proof of Theorem 1.16 is actually another inequality.

Lemma 1.17 (Loomis-Whitney Inequality)**.** *For $d \geq 2$, $j = 1, \ldots, d$, and $f_j = f_j(x^1, \ldots, \widehat{x^j}, \ldots, x^d)$, we have:*

$$|| \prod_{j=1}^{d} f_j ||_{L^1(\mathbb{R}^d)} \leq \prod_{j=1}^{d} ||f_j||_{L^{d-1}(\mathbb{R}^{d-1})}$$

Proof. The proof of Lemma 1.17 is pretty direct, we just integrate in each direction, applying Hölder's inequality as we go.

$$\int |\prod_{j=1}^{d} f_j| dx^1 = |f_1| \int \prod_{j\neq 1}^{d} |f_j dx^1 \leq |f_1| \prod_{j\neq 1}^{d} ||f_j||_{L^{d-1}_{x^1}}$$

Doing this d times gives

$$\int \cdots \int |\prod_{j=1}^{d} f_j| dx^1 \cdots dx^d \leq \prod_{j=1}^{d} ||f_j||_{L^{d-1}_{x^1, \ldots, \widehat{x_j}, \ldots, x^d}}$$

□

Remark. The Loomis-Whitney inequality also has a fun geometric interpretation. Specifically, take $E \subseteq \mathbb{R}^d$, and $\pi_j(E) = \{x \in \mathbb{R}^d : x^j = 0 \text{ and } \exists x^{j\prime} \text{ s.t. } (x^1, \ldots, x^{j\prime}, \ldots, x^d) \in E\}$. The Loomis-Whitney inequality gives us a bound on the measure $\mu(E)$ by the coordinate measures of $\pi_j(E)$. To see this, we take $\mathbf{1}_U$ to be the indicator function of U, and write

$$\mu(E) = \int_{\mathbb{R}^d} \mathbf{1}_E d\mu \leq \int_{\mathbb{R}^{d-1}} \prod_{j=1}^{d} \mathbf{1}_{\pi_j(E)}(x^1, \ldots, \widehat{x^j}, \ldots, x^d) d\mu \leq \prod_{j=1}^{d} \mu(\pi_j(E))^{1/(d-1)}$$

With the Loomis-Whitney inequality (1.17) in mind, we are now ready to prove the Gagliardo-Nirenberg-Sobolev inequality (1.16)!

Proof. (of Theorem 1.16). The first step is to pick $x \in \mathbb{R}^d$, and write $u(x)$ as the indefinite integral

[3] I absolutely did not expect to see dimensional analysis show up anywhere *near* graduate mathematics, and am equal parts happy and confused.

along a coordinate direction

$$u(x) = \int_{-\infty}^{x^j} \partial_{y^j} u(x^1, \ldots, y, \ldots, x^d) dy$$

Since we are free to choose j, we can bound $|u|$ by it's gradient, and define $\tilde{f}_j$ as being equal to the rightmost term on the inequality:

$$|u(x)| \leq \int_{-\infty}^{x^j} |Du(x^1, \ldots, y, \ldots, x^d)| dy \leq \int_{-\infty}^{\infty} |Du(x^1, \ldots, y, \ldots, x^d) dy := \tilde{f}_j$$

Thus, we have that $|u(x)| \leq \left(\prod_{j=1}^d \tilde{f}_j\right)^{1/d}$, and after exponentiating, $|u(x)|^{d/(d-1)} \leq \prod_{j=1}^d \tilde{f}_j^{1/(d-1)}$. Now, we apply the Loomis-Whitney inequality, setting $f_j = \tilde{f}_j^{1/(d-1)}$ to obtain:

$$\begin{aligned}
||u||_{L^{\frac{d}{d-1}}}^{\frac{d}{d-1}} \leq \int |u|^{\frac{d}{d-1}} \leq \int \prod_{j=1}^d f_j^{d-1} dx \leq \prod_{j=1}^d ||f_j||_{L^{d-1}} &= \prod_{j=1}^d \left(\int |f_j|^{d-1} dx^1 \cdots \widehat{dx^j} \cdots dx^d \right)^{\frac{1}{d-1}} \\
&= \prod_{j=1}^d \left(\int \left[\int |Du| dx^j \right] dx^1 \cdots \widehat{dx^j} \cdots dx^d \right)^{\frac{1}{d-1}} \\
&\leq ||Du||_{L^1}^{\frac{d}{d-1}}
\end{aligned}$$

Cancelling exponents on all terms completes the proof. □

Remark. This is a functional version of the Isoperimetric Inequality. One can see this by approximating u as the indicator function of some set.

Another natural question to ask is what happens when varying p on the RHS of Theorem 1.16? This should net an inequaltiy of the form $||u||_{L^q} \leq ||Du||_{L^p}$. Of course, both terms should have the same dimensionality, so we have an equation of the form $[x]^{d/q} \sim [x^{-1+d/p}]$, yielding that $q = \frac{dp}{d-p}$. Stated formally, we have the following theorem:

Theorem 1.18 (Sobolev Inequality for L^p-based spaces)**.** *For $d \geq 2$, $1 \leq p < d$, and $u \in C_0^\infty(\mathbb{R}^d)$, the following inequality holds:*

$$||u||_{L^{\frac{dp}{d-p}}(\mathbb{R}^d)} \leq ||Du||_{L^p(\mathbb{R}^d)} \tag{2}$$

Proof. Define $q = \frac{dp}{d-p}$, $\tilde{q} = q\frac{d-1}{d}$, and $v = |u|^{\tilde{q}}$. The proof uses the following 'napkin-math' calculation: $|Dv| = \tilde{q}|u|^{\tilde{q}-1}|Du|$. This can be justified by using the approximation $|x| = \lim_{\epsilon \to 0^+}(\epsilon^2 + x^2)^{1/2}$, and applying the Dominated Convergence Theorem to integrals where it appears. Then, we may apply Theorem 1.16 to obtain the following inequality:

$$\int |u|^q dx = \int |v|^{\frac{d}{d-1}} \leq \left[\int |Dv| dx\right]^{\frac{d-1}{d}} = \left[\int \tilde{q}|u|^{\tilde{q}-1}|Du| dx\right]^{\frac{d-1}{d}}$$

Now, we apply Hölder's inequality to put $|Du|$ in L^p. A simple dimensional analysis argument gives that the following inequality is the only option

$$\left[\int \tilde{q}|u|^{\tilde{q}-1}|Du|dx\right]^{\frac{d-1}{d}} \leq ||u||_{L^q}^{\frac{d-1}{d}(q-1)}||Du||_{L^p}^{\frac{d-1}{d}}$$

Canceling exponents and dividing factors of u completes the proof. □

The theorems proven above are only for a dense subset of general sobolev spaces. However, they extend naturally to the whole space after an applying our density and extension theorems. We collect those results here:

Theorem 1.19. *Let $d \geq 2$, $1 \leq p < d$, and U be a bounded domain in $\mathbb{R}^d$. Then the following hold:*

i. If $u \in W^{1,p}(\mathbb{R}^d)$ then Equation 2 holds.

ii. If $u \in W_0^{1,p}(U)$, then Equation 2 holds.

iii. If $u \in W_0^{1,p}(U)$, and ∂U is C^1, then

$$||u||_{L^{\frac{dp}{d-p}}(U)} \leq c||u||_{W^{1,p}(U)}$$

Proof. Statements (i) and (ii) follow from an application of the density theorems of the previous section. Statement (iii) requires density and extension. □

Remark. Statement (ii) is sometimes called a Poincaré-type inequality. For statement (iii), we need the full $W^{1,p}$ norm in order to apply the extension procedure. The other two statements come with their own built-in control of the boundary.

1.4.2 $p \geq d$

For completeness, we should consider the case where $p \geq d$. As it turns out, we actually need a different method of relating u to Du, which involves an averaging procedure over bounded balls. We collect this result in the following lemma.

Lemma 1.20. *Let $u \in C^\infty(\mathbb{R}^d)$. Then*

$$\frac{1}{|B_r|}\int_{B_r(x)} |u(x) - u(y)|dy \lesssim \int_{B_r(x)} \frac{|Du|}{|x-y|^{d-1}}dy$$

Proof. We begin by noting that

$$|u(x) - u(y)| = \int_0^1 |\frac{d}{ds}u(x + s(y-x))|ds$$

Then, we take an average over a ball

$$\frac{1}{|B_r|}\int_{B_r(x)} |u(x)-u(y)|dy \leq \frac{1}{|B_r|}\int_{B_r(x)}\int_0^1 |\frac{d}{ds}u(x+s(y-x))|dsdy$$

Taking the integrand out of the right hand side, we have $\partial_s u(x+s(y-x)) = (y-x)\cdot Du(x+s(y-x))$, which allows us to rewrite the previous inequality as

$$\frac{1}{|B_r|}\int_{B_r(x)} |u(x)-u(y)|dy \leq \frac{c}{r^d}\int_{B_r(x)}\int_0^1 |x-y||Du(x+s(y-x))|dsdy$$

Using the polar integration formula, with $\rho\omega = y-x$, $\rho = |y-x|$, we find

$$= \frac{c}{r^d}\int_0^r\int_{\mathbb{S}^{d-1}}\int_0^1 \rho|Du(x+s\rho\omega)|ds\rho^{d-1}d\omega d\rho$$

Changing variables to $t = s\rho$, we find

$$\int_0^r \rho^d|Du(x+s\rho\omega)d\rho = \int_0^{sr}\frac{1}{s}\frac{t^d}{s^d}|Du(x+t\omega)|dt$$

So, we can compute

$$\int_0^1\int_0^{sr}\frac{1}{s}\frac{t^d}{s^d}|Du(x+t\omega)|dtds = \int_0^r\int_{t/r}^1 \frac{1}{s^{d+1}}t^d|Du(x+t\omega)|dsdt \leq c\int_0^r r^d|Du(x+t\omega)|dt$$

Then, the entire inequality is bounded by

$$\begin{aligned}
\frac{1}{|B_r|}\int_{B_r(x)} |u(x)-u(y)|dy &\leq c\frac{1}{\cancel{r^d}}\int_{\mathbb{S}^{d-1}}\int_0^r \cancel{r^d}|Du(x+t\omega)|dtd\omega \\
&= c\int_{\mathbb{S}^{d-1}}\int_0^r \frac{1}{t^{d-1}}|Du(x+t\omega)|t^{d-1}dtd\omega \\
&= c\int_{B_r(x)}\frac{|Du|}{|x-y|^{d-1}}dy
\end{aligned}$$

□

With this result, we may proceed to the case where $p > d$.

Theorem 1.21. *Let $d \geq 2$, $u \in C^\infty(\mathbb{R}^d)$, $p > d$, and set $\alpha = 1 - \frac{d}{p}$. Then*

$$|u(x)-u(y)| \leq c|x-y|^\alpha||Du||_{L^p(\mathbb{R}^d)}$$

Proof. Using the previous lemma, take averages

$$\fint_{B_r(x)} |u(x)-u(z)|dz \leq c\int_{B_r}\frac{|Du|}{|x-z|^{d-1}}dz$$

Now, choose U to such that it is contained in both $B_r(x)$ and $B_r(y)$, and compute:

$$\fint_U |u(x)-u(y)|dz \leq \fint_U |u(x)-u(z)|dz + \fint_U |u(y)-u(z)|dz$$

Note that because of the way we chose U, we also have the inequality

$$\begin{aligned}
\fint_U |u(x)-u(z)|dz &\leq \frac{|B_r(x)|}{|U|}\fint_{B_r(x)} |u(x)-u(z)|dz \\
&\leq c\int_{B_r(x)} \frac{|Du|}{|x-z|^{d-1}}dz \\
&\leq c||Du||_{L^p(B_r(x))}\left\|\frac{1}{|x-z|^{d-1}}\right\|_{L^{\frac{p}{p-1}}(B_r(x))} \\
&\leq \tilde{c}||Du||_{L^p}
\end{aligned}$$

The final inequality follows from the fact that $||1/|x-z|^{d-1}||_{L^{\frac{p}{p-1}}} \sim r^{\alpha}$, which we can assimilate into the constant c. □

Now, we want to define Hölder continuous spaces, which will allow us to deal more concretely and precisely with L^{∞} cases.

Definition 9 (Hölder Seminorm)**.** Let $u \in C(U)$, where U is (typically) a bounded domain. Then the Hölder Seminorm is defined

$$[u]_{C^{\alpha}(U)} = \sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^{\alpha}}$$

To make the Hölder Seminorm into a norm, we have to deal with the fact that $[\cdot]_{C^{\alpha}}$ is invariant under the addition of a constant. To that end, we use the sup norm.

Definition 10 (Hölder Norm)**.** Let $u \in C(U)$, where U is (typically) a bounded domain. Then the Hölder Norm is defined

$$||u||_{C^{\alpha}(U)} = [u]_{C^{\alpha}(U)} + ||u||_{L^{\infty}(U)}$$

As one might expect, Hölder Space is defined as

$$C^{\alpha}(U) = \{u \in C(U) : ||u||_{C^{\alpha}(U)} < \infty\}$$

Theorem 1.22 (Morrey's Inequality)**.** *Let $d \geq 2$, $p > d$, $U \subseteq \mathbb{R}^d$ bounded, and ∂U be C^1. Then the following hold:*

i. $u \in W^{1,p}(U) \Rightarrow u \in C^{\alpha}(U)$ *with* $\alpha = 1 - \frac{p}{d}$.

ii. $||u||_{C^{\alpha}(U)} \leq c||u||_{W^{1,p}(U)}$.

Proof. By extension and density, it suffices to check the case $u \in C_0^{\infty}(\mathbb{R}^d)$, $\operatorname{spt} u \subseteq V$,where V is a fixed (independent of u) bounded open set fully containing $\overline{U}$. By Theorem 1.21 $[u]_{C^{\alpha}(V)} \lesssim$

$c||u||_{W^{1,p}(V)}$, since the $W^{1,p}(U)$ norm contains the $||\cdot||_{L^p}$ term appearing in the inequality. So, it remains only to show that $||u||_{L^\infty(V)} \lesssim ||u||_{W^{1,p}(V)}$.

To that end, fix $x \in \operatorname{spt} u$, and approximate $u(x)$ by an average over $B_r(x)$

$$\left| \fint_{B_r(x)} u(x)dz - \fint_{B_r(x)} u(z)dz \right| \leq \fint_{B_r(x)} |u(x) - u(z)|dz \leq \fint_{B_r(x)} \frac{|Du(z)|}{|x-z|^\alpha} dz \leq r^\alpha ||Du||_{L^p(B_r(x))}$$

Now, take $r = 1$. Then,

$$|u(x)| \leq c \left| \int_{B_1(x)} udz \right| + c||Du||_{L^p(B_1(x))} \leq c(||u||_{L^p} + ||Du||_{L^p})$$

And, in the above chain of inequalities, the final one follows from the fact that $c|\int_{B_1(x)} udz| \leq ||u||_{L^1} \leq c||u||_{L^p}$, where $c = ||1||_{L^{p'}}$. □

In the case where $d = p$, $W^{1,p}$ does not embed into L^∞. The standard counterexample to see this is $U = B_1(0) \subset \mathbb{R}^2$, $u(x) = \ln\big(\ln(10 + |x|^{-1})\big)$. A useful substitute for the Hölder norm is the Bounded Mean Oscillation seminorm.

Definition 11 (Bounded Mean Oscillation)**.** Let $u \in L^1_{\text{loc}}(U)$. The **Bounded Mean Oscillation Seminorm** is defined as

$$[u]_{\text{BMO}} = \sup_{r,x_0} \left\{ \fint_{B_r(x_0)} \left| u(z) - \fint_{B_r(x_0)} u(y)dy \right| dz \right\}$$

Worth noting is that the previous theorem's proof uses the fact that Hölder spaces are complete, which is left as an excercise for a real analysis class. It's left as an excercise to check that $L^\infty \subsetneq \text{BMO}$. e.g. one can use $u = \ln(x)$ on the unit ball. Then, we have the following theorem.

Theorem 1.23. *Let $u \in W^{1,d}(\mathbb{R}^d)$. Then $[u]_{\text{BMO}} < \infty$, and*

$$[u]_{\text{BMO}} \leq c||Du||_{L^d}$$

Proof. Let $u \in C^\infty(\mathbb{R}^d)$, and fix $B_r(x)$. Then we want to show that we can find a c independent of $B_r(x)$ and u such that the inequality above holds. So, we write

$$\begin{aligned}\fint_{B_r(x)} \left| \fint_{B_r(x)} u(z)dy - \fint u(y)dy \right| dz &\leq \frac{1}{|B_r|^2} \int_{B_r(x)} \int_{B_r(x)} |u(z) - u(y)|dydz \\ &\leq \frac{1}{|B_r|^2} \int_{B_r(x)} \int_{B_{2r}(x)} |u(z) - u(y)|dydz \\ &\leq \frac{1}{|B_r|^2} \int_{B_r(x)}^{B_{2r}(x)} \frac{|Du(x)|}{|z-y|^{d-1}} dydz\end{aligned}$$

We will define $F(y) = \int_{B_{2r}(x)} \frac{|Du(z)|}{|z-y|^{d-1}}$, and apply the Hardy-Littlewood Maximal Theorem, with $\mathcal{M}u$

denoting the Hardy-Littlewood Maximal Function.[4] To find a maximal function which controls $F(y)$, we will use a dyadic decomposition. Specifically, $|y|^\alpha$ has the property that if $2^{k-1} \le |y|, |y'| \le 2^k$, then $|y| \simeq |y'| \simeq 2^{k\alpha}$. We also define the balls $A^k = \{2^{k-1} \le |z-y| \le 2^k\}$. Using our new decomposition, we write

$$\begin{aligned}\int_{B_{2r}(y)} \frac{|Du|}{|z-y|^{d-1}} dz &\le \sum_{2^k \le 2r} \int_{A^k} \frac{1}{(2^k)^{d-1}} |Du(z)| dz \\ &\le \sum_{2^k \le 2r} \frac{1}{(2^k)^{d-1}} \int_{B_{2^k}(y)} |Du| dz \\ &\le c \sum_{2^k \le 2r + c} 2^k [\mathcal{M}|Du|](y)\end{aligned}$$

Then, we take an L^1 norm, and apply Hölder. The sum drops out since it's geometric and of size r, we assimilate it into the constant c.

$$\begin{aligned}||c \sum_{2^k \le 2r+c} 2^k [\mathcal{M}|Du|](y)||_{L^1} &\le cr||\mathcal{M}|Du|||_{L^1} \\ &\le cr||\mathcal{M}|Du|||_{L^d(B_r(x))}||1||_{L^{\frac{d}{d-1}}(B_r(x))} \\ &\le cr||\mathcal{M}|Du|||_{L^d(B_r(x))} r^{d-1} \\ &\le cr^d||Du||_{L^d}\end{aligned}$$

The final inequality is an application of Hardy-Littlewood. When we move cr^d over to the left-hand side of the inequality, we get the desired result, since it satisfies the averaging condition laid out in the theorem statement. □

1.5 Compactness Theorems

Definition 12 (Compact Operator). Let $T : X \to Y$ be a bounded, linear operator between normed spaces X, Y. T is called a **compact operator** if either of the equivalent statements are true.

- $T(B_X)$ is compact in Y.
- For all bounded sequences $\{x_n\} \subset X$, $\{T(x_n)\}$ has a convergent subsequence in Y.

Definition 13. Suppose $i : X \hookrightarrow Y$ is linear ($\hookrightarrow$ denotes that i is an embedding). Then $X \subseteq Y$ has a natural identification and is compact if i is compact.

These are the definitions involved in the natural setting of compactness, where we want to examine for which values of q, p $W^{1,p}(U) \subseteq L^q(U)$. The basic compactness theorem from functional analysis is the Arzelá-Ascoli theorem, which we recall below.

[4] One might be tempted to apply Young's Inequality here, since F is clearly a convolution. This will fail, however, since $\frac{1}{|z-y|^{d-1}} \notin L^q$ for the q we will need from dimensional analysis.

Theorem 1.24 (Arzelà-Ascoli). *Let K be a compact Hausdorff space, and $F \subseteq C(K)$, where $C(K)$ is equipped with the uniform topology. Then F is compact if and only if the following hold*

- *(Local Boundedness) For all $x \in K$, there exists M_x such that for all $f \in F$, $|f(x)| \leq M_x$.*
- *(Equicontinuity) For all $\epsilon > 0$, there exists $\delta > 0$ such that for all $f \in F$, $|f(x) - f(y)| < \epsilon$ if $|x - y| < \delta$.*

Theorem 1.25 (Compactness of $C^{0,\alpha}(U) \subseteq C^{0,\alpha'}(U)$). *Let $U \subset \mathbb{R}^d$ be a bounded, open set, and let $0 < \alpha' < \alpha < 1$. Then $C^{0,\alpha}(U) \subset C^{0,\alpha'}(U)$ is compact.*

A sketch of the proof was given in lecture. I will fill in details.

. The basic steps are (i) note that by a direct application of Arzelà-Ascoli, $C^{0,\alpha}(U) \subseteq C(U)$ is compact. (ii) Show that bounded sequences in the Holder space have convergent subsequences in $C(U)$. (iii) show that $||u_{n_j} - u_\infty||_{C^{0,\alpha'}} \to 0$ using interpolation. The L^∞ part goes to 0 by convergence in C^0. The Holder seminorm part uses the following inequality

$$[v]_{C^{0,\alpha'}} \leq c||v||_{L^\infty}^{1-\frac{\alpha'}{\alpha}} [v]_{C^{0,\alpha}}^{\alpha/\alpha}$$

The exponents are derived from a homogeneity argument. □

Theorem 1.26 (Rellich-Kondrachov). *Let $d \geq 2$, $U \subseteq \mathbb{R}^{d \geq 2}$ be bounded and open, with $\partial U \in C^1$. Let $1 \leq p < d$, and $1 \leq q < \frac{d}{\frac{d}{p} - 1}$. Then $W^{1,p} \hookrightarrow L^p$ is compact.*

The proof of Theorem 1.26 relies on the following lemma regarding the convergence of mollifiers.

Lemma 1.27 (Accelerated Convergence of Mollifiers). *Suppose $v \in W^{k,p}$, $1 \leq p < \infty$, and $\varphi \in C_0^\infty(\mathbb{R}^d)$ is a mollifier satisfying an additional moment condition: $\int x^\alpha \varphi dx = 0$ for all $1 \leq |\alpha| < k$. Then*

$$||\varphi_\epsilon * v - v||_{L^p} \leq c\epsilon^k ||v||_{W^{k,p}}$$

Proof. We should take the Taylor expansion (using α_k to denote all multi-indices of degree k)

$$v(x-y) - v(x) = \sum_{n=1}^{k-1} \frac{1}{n!} y^{\alpha_n} D^{\alpha_n} v(x) + \frac{1}{k!} \int_0^1 (1-s)^k \partial_s v(x - sy) ds$$

and put it into our convolution

$$\begin{aligned}
\int \varphi_\epsilon(y)(v(x-y)-v(x))dy &= \int \varphi_\epsilon \left(\sum_{n=1}^{k-1} \frac{1}{n!} y^{\alpha_n} D^{\alpha_n} v(x) + \frac{1}{k!}\int_0^1 (1-s)^k \partial_s v(x-sy)ds \right) dy \\
&= \sum_{n=1}^{k-1} \frac{1}{n!} D^{\alpha_n} v(x) \cancelto{0}{\int \varphi_\epsilon(y) y^{\alpha_n} dy} + \frac{1}{k!} \int \int_0^1 \varphi_\epsilon(y)(1-s)^k \partial_s v(x-sy) ds dy \\
&= \frac{1}{k!} \int \int_0^1 \varphi_\epsilon(y)(1-s)^k \partial_s v(x-sy) ds dy \\
&\simeq \int \varphi_\epsilon(y) \int_0^1 \partial_s^{k+1} v(x-sy) ds dy \\
&\simeq \int \varphi_\epsilon(y) y^{\alpha_{k+1}} \int_0^1 D_y^{\alpha_{k+1}} v(x-sy) ds dy
\end{aligned}$$

Now, we consider $|y| \lesssim \epsilon$ on the support of the integral, and note that the convolution can be bounded in L^p by taking the L^p norm of both sides, and applying an inequality. The proof of this went a bit fast in lecture, and I'm not sure I understand it. $\square$

Using the above lemma, we are ready to prove Rellich-Kondrachov.

Proof of 1.26. First, we reduce to the case $W^{1,p} \hookrightarrow L^p$. There are two subcases. The first is when $1 \le q \le p$. Since U is bounded, $||v||_{L^q(U)} \le |U|^{\frac{1}{q}-\frac{1}{p}} ||v||_{L^p(\mathbb{R}^d)}$ by Hölder's inequality. The second is $p < q < p^*$. In this case $||v||_{L^q(U)} \le ||v||^{\theta}_{L^p(U)} ||v||^{1-\theta}_{L^{p^*}(U)}$, where by dimensional analysis, θ must satisfy $\frac{d}{q} = \frac{d}{p}\theta + \frac{d}{p^*}(1-\theta)$. Then by the previous analysis, $\theta < p$, so $||v||^{\theta}_{L^p} \to 0$, and the bound on $||v||_{L^{p^*}}$ is given by the sobolev inequalities shown previously.

Now, it suffices to consider the case $q = p$ and apply the previous lemma in leiu of equicontinuity. Given $\{u_n\} \subset W^{1,p}$ and bounded by some real M, we can, by the extension theorem, find a sequence $\{\tilde{u}_n\}$ extending u_n defined on the whole space, with $\operatorname{spt} \tilde{u}_n \subseteq V$, where V is a bounded open set containing U. Furthermore, we have the bound

$$||\tilde{u}_n||_{W^{1,p}(\mathbb{R}^d)} \le c||u_n||_{W^{1,p}(U)} \le cM$$

Now, introduce the mollifier

$$\tilde{u}_n = \cancelto{v_{n,\epsilon}}{\varphi_\epsilon * \tilde{u}_n} + \cancelto{e_{n,\epsilon}}{(\tilde{u}_n - \varphi_\epsilon * \tilde{u}_n)}$$

$v_{n,\epsilon}$ is smooth, and higher-regularity norms are bounded. By the lemma, the error term uniformly converges: $||e_{n,\epsilon}||_{L^p(\mathbb{R}^d)} \le c\epsilon M$. Furthermore, applying Hölder yields that

$$||v_{n,\epsilon}||_{L^\infty} + ||\nabla v_{n,\epsilon}||_{L^\infty} \le c_\epsilon M$$

So, by Arzelá-Ascoli, for each ℓ, there exists a subsequence $\tilde{u}_{n_\ell}$ such that $||e_{n_\ell,\epsilon}||_{L^\infty} \le 2^{-\ell}$, and $||v_{n_{\ell'}} - v_{n_{\ell''}}||_{L^p} \le 2^{-\ell}$ for $\ell', \ell'' \ge \ell$. Now, all that remains is a diagnolization argument and to extend the sequence, which completes the proof. $\square$

1.6 Poincaré-Type Inequalities

Loosely speaking, a Poincaré-Type inequality is any inequality controlling a function using information about the derivative, with a condition fixing the ambiguity introduced by the addition of a constant under integration.

Theorem 1.28 (Poincaré Inequality). *Let $1 \leq p < \infty$, $U \subset \mathbb{R}^d$ bounded with C^1 boundary. Then, for $u \in W^{1,p}(U)$ where $\int_U u dx = 0$,*

$$||u||_{L^p} \leq c_U ||Du||_{L^p}$$

Proof. There are a few proofs of this theorem, ours uses compactness and an argument by contradiction. Assume such a c_U doesn't exist, i.e. for all $n \geq 1$, there exists $||u_n||_{L^p} \geq n||Du||_{L^p}$, and $\int_U u_n dx = 0$. By normalization, we may assume that $||u_n||_{L^p(U)} = 1$. Then it must be that $||Du||_{L^p} \leq \frac{1}{n}$, and therefore, $||u||_{W^{1,p}(U)} \leq 2$.

By the Rellich-Kondrachov theorem, however there must exist an L^p-convergent subsequence of u_n, implying that $||u_n||_{L^p} = ||u_\infty||_{L^p} = 1$. Introducing $\varphi \in C_0^\infty(U)$, we note also that there must exist a subsequence u_{n_j} converging to u. In this case then,

$$\int_U u_{n_j} \partial_k \varphi dx = -\int_U \partial_k (u_{n_j}) \varphi dx = 0$$

since that norm goes as $1/n_j$. Thus, Du is 0 almost everwhere, and u is a constant. This contradicts that $||u||_{L^p} = 1$, so we are done. □

There are many more such inequalities, which were stated without proof.

- Freiderich Inequality: the boundary condition is instead $u_| \partial U = 0$. This can be proven using both compactness, and the sobolev inequality for $W_0^{1,p}$.
- Hardy's inequality comes in two forms.
  - If $u \in W^{1,p}(U)$ with $u'|_{\partial U} = 0$, then

$$||\frac{1}{\text{dist}(\partial U, x)} u(x)||_{L^p} \leq c||Du||_{L^p}$$

  - If $u \in W_0^{1,p}$, with $p < d$, then

$$||\frac{1}{|x|} u(x)||_{L^p} \leq c||Du||_{L^p}$$

2 Linear Elliptic PDEs

2.1 Introduction to Linear Elliptic PDEs

Heuristically speaking, elliptic partial differential equations are generalizations of Laplace's equation $-\Delta u = f$. In particular, the sum $-\sum_j \partial_i \partial_j \mapsto -\sum_j (-\xi_j)^2 = |\xi|^2$ in Fourier space, and since $|\xi| \neq 0$, elliptic PDEs are invertible in fourier space. This transformation of the coefficients of the leading-order operator is called the *principal symbol* of the operator.

Definition 14 (Elliptic Operators)**.** A partial differential operator P is elliptic if the principal symbol of P is invertible for all x in its domain, and $\xi \neq 0$.

An important subcase of this definition is the case where u is a scalar function. In this case, the ellipticity condition holds if and only if the coefficient a^{ij} in the operator $a^{ij}\partial_i\partial_j$ is definite. We will assume positive definiteness for simplicity. Furthermore, we define *uniform ellipticity*.

Definition 15 (Uniformly Elliptic Scalar Operator)**.** An elliptic operator $P = a^{ij}\partial_i\partial_j + \ldots$ is *uniformly elliptic* if $\exists \lambda > 0$ such that $a^{ij}\xi_i\xi_j \geq \lambda$ for all $|\xi| = 1$. Equivalently, the eigenvalues of a^{ij} must be bounded from below.

Elliptic PDEs arise in a variety of applications. For example

- Methods in the calculus of variations often yield elliptic PDEs as optimizers.
- Evolutionary problems such as the incompressible Euler equation, with $u : \mathbb{R}^{3+1} \to \mathbb{R}^3$ the velocity field of a fluid element, with pressure ρ, and incompressibility condition $\nabla u = 0$.

$$\partial_t u + u \cdot \nabla u + \nabla \rho = 0$$

In this case, incompressibility determines ρ, which we can see

$$\nabla(\cancelto{0}{\partial_t u} + u \cdot \nabla u + \nabla \rho) = 0 \Rightarrow -\Delta \rho = \nabla(u \cdot \nabla u)$$

The final expression is an elliptic PDE for ρ.

2.2 Boundary Value Problems

For the remainder of this section, we assume the scalar case in $\mathbb{R}^{d \geq 2}$, with P a uniformly elliptic PDO with sufficiently regular coefficients. U is a bounded, open, connected domain in $\mathbb{R}^d$, with sufficiently "nice" boundary. In particular, we will study the solveability of the Dirichlet boundary value problem in $H^1(U)$, since we will need to use the trace theorem. Other BVP's (for example Neumann) require $u \in H^2$, since they deal with prescribing derivatives on the boundary.

Finally, we also make a standard reduction to the case where $u_{|\partial U} = 0$, which is justified by taking any extension of u beyond U, and subtracting it on the boundary.

Definition 16 (Divergence-Form Operator)**.** A partial differential operator P is of *divergence-form* if it is expressed as

$$Pu = \partial_i(a^{ij}\partial_j u) + \partial_i(b^i u) + cu$$

2.2.1 Uniqueness

Our discussion of uniqueness depends on making an a-priori estimate.

Theorem 2.1. *Suppose $u \in H^1(U)$ solves the Dirichlet problem, with a, b, c sufficiently regular. Then there exist constants c, γ such that*

$$||u||_{H^1(U)} \leq c||Pu||_{H^{-1}(U)} + \gamma||u||_{L^2(U)}$$

Proof. The proof is an excercise in integration by parts of the divergence form.

$$\int_U (Pu)udx = \int_U u(\partial_i(a^{ij}\partial_j u) + \partial_i(b^i u) + cu)dx = \int_U -a^{ij}\partial_i u\partial_j u - b^j u\partial_j u + cu^2 dx$$

The first term in the rightmost expression is bounded by uniform ellipticity: $\lambda|Du|^2 \leq a^{ij}\partial_i u\partial_j u$. Thus, we have $\int_U fudx \leq c||f||_{H^{-1}}||u||_{H^1}$, which gives that (putting the entire expression with lower order terms back in)

$$\lambda||Du||^2_{L^2} \leq c||f||_{H^{-1}}||u||_{H^1} + \int_U |b||\partial u||u|dx + \int_U |c||u|^2 dx$$

If we put $b, c \in L^\infty$, they can be factored out of the integral, which yields the desired bounds. We can get optimal regularity for $b \in L^{d+}$ and $c \in L^{d/2+}$, but the discussion here was brief. □

2.2.2 Existence

Missed this lecture due to being stuck in the Detroit airport. Nice.

2.3 Elliptic Regularity

As a prototypical example, suppose we have $f \in H^k$ (or more generally, $C^{k,\alpha}$) and $-\Delta u = f$ in U. Then for all $V \subseteq\subseteq U$, heuristically, we expect u to be more regular than f by 2 derivatives. This is the so-called Elliptic Interior Regularity. We begin with a discussion of interior and boundary regularity in L^2, before moving on to L^∞-based estimates (so-called Schauder estimates).

2.3.1 L^2-Based Regularity

All of our L^2 regularity theorems take place in Hilbert space. The prototypical elliptic PDE is *Poisson's Equation* $-\Delta u = f$. As an illustration of the proof techniques used throughout this section, consider $-\Delta u = f$ on a bounded domain $U \subset \mathbb{R}^d$, with $u \in H^1(U)$. Clearly, we can't just integrate by parts on

U, since there will be boundary terms. In order to deal with this, introduce a smooth cutoff function from some set $V \subseteq\subseteq U$ to U^c called ζ, and try an energy method calculation.

$$\begin{aligned}\int_U fu\zeta^2 dx &= \int_U -(\Delta u)u\zeta^2 dx \\ &= \sum_j \int_U (\partial_j u)(\partial_j(u\zeta^2))dx \\ &= \sum_j \int_U (\partial_j u)^2\zeta^2 + 2u\zeta(\partial_j u)(\partial_j \zeta)dx \\ &= \sum_j \int_U (\partial_j u)^2(\zeta)^2 + 2u\zeta(\partial_j u)(\partial_j \zeta)dx\end{aligned}$$

Rearranging, and applying Cauchy-Schwarz to the right-hand side, we have that

$$\int_U |Du|^2\zeta^2 dx \leq \left|\int_U fu\zeta^2 dx\right| + 2\left|\int_U u\zeta(Du)\cdot(D\zeta)dx\right|$$

Now, use Hölder's inequality and the identity $\frac{1}{\sqrt{\epsilon}}a\sqrt{\epsilon}b = ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, on the rightmost term to find

$$2\left|\int_U (u\zeta)(Du\cdot D\zeta)dx\right| \leq \left(2\int_U (|Du|\zeta)^2 dx\right)^{1/2}\left(\int_U (u|D\zeta|)^2 dx\right)^{1/2} \leq \epsilon\int_U |Du|^2\zeta^2 dx + \frac{1}{\epsilon}\int_U u^2|D\zeta|^2 dx$$

Choosing $\epsilon = 1/2$, and substituting back in, we find that

$$\frac{1}{2}\int_U |Du|^2\zeta^2 dx \leq \left|\int_U fu\zeta^2 dx\right| + 2\int_U u^2|D\zeta|^2 dx$$

Note that if we assume u to be compactly supported in U, we may set the rightmost integral equal to zero, since ζ can be chosen to be constant on $\operatorname{spt} u$. Under this assumption, applying Hölder gives the estimate

$$||Du||_{L^2} \leq 2||f||_{L^2} + 2||u||_{L^2}$$

As we will see, we want our right hand side to be in H^{-1}, which means we should have introduced a rougher (H^{-1}) cutoff function ζ.

In order to secure higher-order regularity, we can commute derivatives with our integral equation above, integrate by parts a bunch, and retrieve the result.

Interior Regularity For the following section, we let

$$Pu = -\partial_j(a^{jk}\partial_k u) + b^j\partial_j u + cu \tag{3}$$

with $u : U \to \mathbb{R}$ where U is open in $\mathbb{R}^d$. The coefficients a, b, c are all in $L^\infty(U)$. Furthermore, a is uniformly elliptic, i.e. there exists $\lambda > 0$ such that $\forall x \in U$, $|a(x)| \geq \lambda$. $Da \in L^\infty(U)$ as well.

For the following proof, we need Difference Quotients.

Definition 17 (Difference Quotients)**.** Let e_k denote the k^{th} unit vector, and $h \in \mathbb{R}$. Then for a function u on $\mathbb{R}^d$, $D_k^h u := \frac{u(x+he_k)-u(x)}{h}$.

Also worth noting is the identity $D_j^h(u \cdot v) = D_j^h(u(x))v(x) + u(x+h)D_j^h v(x)$, and the following estimate regarding difference quotients.

Lemma 2.2. *Let $V \subseteq\subseteq U$ and $u \in W^{1,p}(U)$. Then if $|h| < \frac{1}{2}\operatorname{dist}(V, \partial U)$,*

$$||D^h u||_{L^p(V)} \le C||Du||_{L^p(U)}$$

If $1 < p < \infty$, $u \in L^p(V)$m and there exists C such that $||D^h u||_{L^p(V)} \le C$, then $u \in W^{1,p}(V)$ with $||Du||_{L^p(V)} \le C$.

Proof. We can use the fundamental theorem of calculus to get an estimate.

$$\begin{aligned}
u(x+he_k) - u(x) &= \int_0^1 \partial_k u(x+the_k)he_k dt \\
|u(x+he_k) - u(x)| &\le h\int_0^1 |\partial_k u(x+the_k)|dt \\
\int_V |u(x+he_k) - u(x)|^p dx &\le C\sum_{k=1}^d \int_V \int_0^1 |Du(x+the_k)|^p dtdx \\
&= \sum_{k=1}^d \int_0^1 \int_V |Du(x+the_k)|dxdt
\end{aligned}$$

The rest follows by approximation and applying the fundamental theorem. □

Theorem 2.3 (H^2 Elliptic Interior Regularity)**.** *Let $u \in H^1$ be a weak solution to $Pu = f$ on U, with $f \in L^2(U)$, $a \in C^1(U)$, and $b, c \in L^\infty(U)$.*[5] *Then $\forall V \subseteq\subseteq U$, u is in $H^2(V)$, and*

$$||u||_{H^2(V)} \le C\left(||f||_{L^2(U)} + ||u||_{L^2(U)}\right)$$

Proof. To begin, let ζ be a smooth cutoff function from V to U^c. Furthermore, define $g = f - b^j\partial_j u - cu$, noting that since u is a weak (H^1) solution to $Pu = f$, $\langle Pu, \varphi\rangle = \langle f, \varphi\rangle$ for ever $\varphi \in H_0^1(U)$. This works, since $f \in L^2 \subseteq H^{-1}$, and the result we want to show is $Pu \in H^{1-2} = H^{-1}$, $H_0^1 = (H^{-1})^*$ is the appropriate dual pairing φ. Using our definition of g, and letting $\tilde{P}u = Pu - g$ an equivalent starting point is the following:

$$\langle \tilde{P}u, \varphi\rangle = \langle g, \varphi\rangle$$

[5] My treatment here follows Evan's text more closely than lecture. I was having a hard time understanding where some minus signs came from and decided the text proved a sharper result anyhow.

Our first goal is to commute this equation with a difference quotient. Letting φ be a test function,

$$\begin{aligned}\langle D_l^h \tilde{P}u, \varphi\rangle &= \int -D_l^h(\partial_j(a^{jk}\partial_k u))\varphi dx \\ &= \int D_l^h(a^{jk}\partial_k u)\partial_j \varphi dx \\ &= \int \partial_j(a^{jk}\partial_k u)(D_l^{-h}\varphi)dx \\ &= \langle \tilde{P}u, -D_l^{-h}\varphi\rangle\end{aligned}$$

Given that we derived the H^1-estimate by choosing $\varphi = u\zeta^2$ in the case of Poisson's equation, a good place to start might be by choosing one higher derivative of u in the test function, i.e. something like $\varphi = |Du|\zeta$. Of course, using this approach is going to put too many derivatives on u, and of course we can no longer make use of the fact that $u \in H^1$. With this in mind, we instead are going to choose our test function to be $\varphi = \zeta^2 D_l^h u$, and examine limiting behavior.

With our commutation argument, we should evaluate the following:

$$\begin{aligned}\int_U -gD_l^{-h}(\zeta^2 D_l^h u)dx &= \int_U \partial_j(a^{jk}\partial_k u)D_l^{-h}(\zeta^2 D_l^h u)dx \\ &= \int_U -(a^{jk}\partial_k u)D_l^{-h}(\partial_j(\zeta^2 D_l^h u))dx \\ &= \int_U D_l^h(a^{jk}\partial_k u)(\partial_j(\zeta^2 D_l^h u))dx \\ &= \int_U (a^{jk}(x+h)D_l^h\partial_k u + \partial_k u D_l^h a^{jk})(2D_l^h u\zeta\partial_j\zeta + \zeta^2 D_l^h \partial_j u)dx \\ &= A + B\end{aligned}$$

Here, we define

$$A = \int_U a^{jk}(x+h)\zeta^2 D_l^h\partial_k u D_l^h \partial_j u dx$$

$$B = \int_U \left[2(D_l^h u)a^{jk}(x+h)\zeta(D_l^h\partial_k u)(\partial_j\zeta) + 2\zeta(D_l^h u)(D_l^h a^{jk}\partial_j\zeta\partial_k u) + \zeta^2(D_l^h a^{jk})(D_l^h\partial_j u)\partial_k u\right] dx$$

We can get a lower bound on A by applying the uniform ellipticity condition:

$$A \geq \lambda \int_U |D_l^h Du|^2\zeta^2 dx$$

For B, we use that fact that a, b, c are all bounded[6], as is the derivative of ζ[7] to deduce that

$$|B| \leq C\int_U \zeta|D_l^h Du||D_l^h u| + \zeta|D_l^h u||Du| + \zeta|D_l^h Du||Du|dx$$

[6] This isn't technically implied by the C^1-continuity of a^{jk}, but I think it has to be true to justify this leap.
[7] Urysohn's lemma on $\mathbb{R}$ guarantees this.

Applying Cauchy's inequality ($ab \leq 1/2(a^2 + b^2)$) twice yields the further estimate

$$|B| \leq \epsilon \int_U \zeta^2 |D_l^h Du|^2 dx + \frac{C}{\epsilon} \int_U |D_k^h u|^2 + |Du|^2 dx$$

Using the estimate that for $W \subseteq\subseteq U$ but completely contains V , $||D^h u||_{L^p(W)} \lesssim ||Du||_{L^p(U)}$ (evans 5.8.2), we can further restrict this estimate

$$|B| \leq \epsilon \int_U \zeta^2 |D_l^h Du|^2 dx + \frac{2C}{\epsilon} \int_U |Du|^2 dx$$

Picking appropriate $\epsilon = \lambda/2$, we can subtract $|B|$ from A to obtain the lower bound

$$A + B \geq \frac{\lambda}{2} \int_U |D_l^h Du|^2 \zeta^2 dx - C \int_U |Du|^2 dx \tag{4}$$

Now, we wish to place an estimate on the other side of the equation, namely $\langle g, \varphi \rangle$. Factoring out constants yields the estimate

$$\left| \int_U g\varphi dx \right| \leq C \int (|f| + |Du| + |u|)(|\varphi|) dx \tag{5}$$

Using the same estimate from Evans 5.8.2 as above, we can deduce an estimate on $\int_U |\varphi|^2$, and then apply Cauchy's inequality. Choosing $V \subseteq\subseteq W \subseteq\subseteq U$ appropriately, we can estimate

$$\begin{aligned}
\int_U |\varphi|^2 dx &\leq C \int_U |D(\zeta^2 D_l^h u)|^2 dx \\
&\leq C \int_W |D_l^h u|^2 + \zeta^2 |D_l^h Du|^2 dx \\
&\leq C \int_U |Du|^2 + \zeta^2 |D_l^h Du|^2 dx
\end{aligned}$$

Now apply Cauchy's inequality to the individual terms of Equation 5 to obtain the estimate

$$\left| \int_U g\varphi dx \right| \leq \epsilon \int_U \zeta^2 |D_l^h Du|^2 dx + \frac{C}{\epsilon} \left(\int_U f^2 + u^2 + |Du|^2 dx \right) \tag{6}$$

With a choice of $\epsilon = \lambda/4$, we can combine Equations 6 and 4 to find

$$\int_V |D_l^h Du|^2 dx \leq \int_U |D_l^h Du|^2 dx \leq C \int_U |f|^2 + |u|^2 + |Du|^2 dx$$

Thus, by the second half of the estimate of Evans, we have that Du is locally H^1, and so u is locally H^2. Furthermore, by splitting up C appropriately, we may rewrite the above estimate as

$$||u||_{H^2(V)} \leq C(||f||_{L^2(U)} + ||u||_{H^1(U)})$$

□

Theorem 2.4 (H^k Interior Elliptic Regularity)**.** *Let $u \in H^k(U)$ be a weak solution to $Pu = f$ on U, with $f \in H^{k-2}(U)$, $b, c \in L^\infty(U)$, with $|D^\alpha b| + |D^\alpha c|$ bounded for all $|\alpha \leq k-2$, and $a \in C^1(U)$ with $|D^\alpha a|$ bounded for all $|\alpha| \leq k-1$.*

Then for all $V \subseteq\subseteq U$, there exists C such that

$$||u||_{H^k(V)} \leq C(||f||_{H^{k-2}(U)} + ||u||_{L^2(U)})$$

Proof. The proof uses induction on k, with $k = 2$ being the base case of the previous theorem. The full proof can be found in Evans text. □

Theorem 2.5 ($H^2 Boundary Regularity$)**.** *Let $u \in H^1_0(U)$ be a weak solution to $Pu = f$ on U, with $f \in L^2(U)$, $a \in C^1(U)$, $b, c \in L^\infty(U)$, and ∂U of class C^2. Then for all $V \subseteq\subseteq U$, $u \in H^2(V)$, and*

$$||u||_{H^2(V)} \leq C(||f||_{L^2(U)} + ||u||_{L^2(U)})$$

.

Proof. In order to prove this, let's apply the method from Theorem 2.3 and see what goes wrong. Omitting the contributions of b, c as before, we find that

$$\partial_l f = -\partial_l \partial_j (a^{jk} \partial_k u)$$

This is fine, with the exception that $\partial_l f$ can't be evaluated transversal to ∂U. With this in mind, our goal is to take that $d-1$ admissible derivatives, and use them to derive an algebraic condition on ∂f which yields a nice regularity result.

Since ∂U is regular enough, it suffices to consider the special case where $\operatorname{spt} u \subset B_{\frac{1}{2}}(0) \cap \mathbb{R}^d_+$, where $U = B_1 \cap \mathbb{R}^d_+$. In thise case, commuting ∂_l for any $1 \leq l \leq d-1$ yields exactly the same proof as the previous case, as the trace assumption gives a boundary term which goes to 0.

Now, we only need to control $||\zeta^2 \partial_d^2 u||_{L^2}$, which we can accomplish via the following argument. Since a^{jk} is uniformly elliptic, $a^{jk}\xi_j\xi_k \geq \lambda|\xi|^2$. Thus, $a^{dd} \geq \lambda$. Separating out the a^{dd} term, we find

$$\begin{aligned} f &== -\partial_d(a^{dd}\partial_d u) - \partial_j(a^{jk}\partial_k u)(1-\delta_{jd})(1-\delta_k d) \\ &= -a^{dd}\partial_d^2 u - (\partial_d a^{dd})\partial_d u - \partial_j(a^{jk}\partial_k u)(1-\delta_{jd})(1-\delta_k d) \end{aligned}$$

Dividing by a^{dd} is legal, since its bounded from below, so we arrive at

$$\partial_d^2 u = \frac{1}{a^{dd}}(f + (\partial_d a^{dd})\partial_d u + \partial_j(a^{jk}\partial_k u)(1-\delta_{jd})(1-\delta_k d))$$

Taking the L^2 norm yields the desired control.

This argument extends to the general case via a partition of unity, as in the proof of extension theorems for Sobolev Spaces. There are a few important caveats.

- We must check each boundary straightening actually straightens the boundary.
- We must check that the ellipticity bound holds in the new variables, as well as the bounds on the derivatives. This holds by conjugating a with the jacobian.
- The C^2 property of ∂U is necessary to undo this boundary straightening, as derivatives of the jacobian appear.

□

2.3.2 Schauder Estimates

This section was pretty short. Professor Oh mentioned he would post notes on Schauder estimates, and then never did, so this is really more of a speedrun through the theorems. We do touch on Littlewood-Payley theory, which comes back to play a big role later.

First, we (re)state the definition of a Hölder Space. Recall the definition of the Hölder Norm (with a small change of notation):

$$||u||_{C^{0,\alpha}U} = [u]_{C^{\alpha}(U)} + ||u||_{L^{\infty}(U)}$$

Definition 18 (Hölder Space)**.** The Hölder Space $C^{k,\gamma}(U)$ is defined as

$$C^{k,\gamma} = \left\{ u : \sum_{|\alpha|\leq k} ||D^{\alpha}u||_{L^{\infty}(U)} + \sum_{|\alpha|=k} [D^{\alpha}u]_{C^{0,\gamma}} < \infty \right\}$$

Here, we just state a bunch of theorems, and list the ideas of proof techniques.

Theorem 2.6 (Divergence form interior Schauder estimate)**.** *Let $U \subseteq \mathbb{R}^d$ be open, with $u \in C^{k,\alpha}(\overline{U})$[8] a solution of $f = -\partial_j(a^{jk}\partial_k u)$, where a is uniformly elliptic everywhere, $a \in C^{k-1,\alpha}(\overline{U})$, $f \in C^{k-2,\alpha}(\overline{U})$, and $0 < \alpha < 1$, $k \geq 1$. Then $\forall V \subseteq\subseteq U$, there exists a constant C_V such that*

$$||u||_{C^{k,\alpha}(V)} \leq C_V(||u||_{C^{0,\alpha}(U)} + ||f||_{C^{k-2,\alpha}(U)})$$

A similar estimate exists for non-divergence equations, with the regularity of a lowered to $C^{k-2,\alpha}$ (since one derivative moves off a.

Theorem 2.7 (Non-divergence form interior Schauder estimate)**.** *Let $U \subseteq \mathbb{R}^d$ be open, with $u \in C^{k,\alpha}(\overline{U})$ a solution of $f = -a^{jk}\partial_j\partial_k u)$, where a is uniformly elliptic everywhere, $a \in C^{k-2,\alpha}(\overline{U})$, $f \in C^{k-2,\alpha}(\overline{U})$, and $0 < \alpha < 1$, $k \geq 1$. Then $\forall V \subseteq\subseteq U$, there exists a constant C_V such that*

$$||u||_{C^{k,\alpha}(V)} \leq C_V(||u||_{C^{0,\alpha}(U)} + ||f||_{C^{k-2,\alpha}(U)})$$

[8] This assumption is an apriori estimate.

Theorem 2.8 (Divergence form boundary Schauder estimate). *Let $U \subseteq \mathbb{R}^d$ be bounded, with $\partial U \in C^{k,\alpha}$. Let $u \in C^{k,\alpha}(\overline{U})$ a solution of $f = -\partial_j(a^{jk}\partial_k u)$, where a is uniformly elliptic everywhere, $a \in C^{k-1,\alpha}(\overline{U})$, $f \in C^{k-2,\alpha}(\overline{U})$, and $0 < \alpha < 1$, $k \geq 1$. Then*

$$||u||_{C^{k,\alpha}(U)} \leq C_V(||u||_{L^0(U)} + ||f||_{C^{k-2,\alpha}(U)})$$

The non-divergence boundary estimate is exactly what you would expect. Overall, the strategy for doing these proofs proceeds in three steps:

1. Obtain the result in the constant coefficient case.
2. Apply the method of freezing coefficients (locally a^{jk} is regular enough to be treated as constant. We did not cover this in lecture.)
3. To get the boundary result, use boundary straightening and a partition of unity to extend the half-ball case everywhere.

We covered two approaches in the constant coefficient case: Littlewood-Payley theory and Compactness + Contradiction.

Corrolary (Constant-Coefficient Interior Regularity). *Let U be open, and u be a solution to $-\partial_j a^{jk}\partial_k u = -a^{jk}\partial_j\partial_k u = f$ where a is constant and elliptic on $\mathbb{R}^d$. For $u \in C_0^{k,\alpha}(\mathbb{R}^d)$, $f \in C^{k-2,\alpha}(\mathbb{R}^d)$,*

$$||u||_{C^{k,\alpha}(\mathbb{R}^d)} \leq c(||f||_{C^{k-2,\alpha}(\mathbb{R}^d)})$$

Note that the compact support condition of u gets rid of the C^0 term.

Definition 19 (Littlewood-Paley Projection). Define the cutoff function

$$\chi_{\leq 0}(\xi) = \begin{cases} 1 & |\xi| \leq 1 \\ 0 & |\xi| \geq 2 \end{cases}$$

Call $\chi_{\leq k}(\xi) = \chi_{\leq 0}\left(\frac{\xi}{2^k}\right)$. Then we define $\chi_k = \chi_{\leq k+1} - \chi_{\leq k}$, which is a smooth cutoff function with $\operatorname{spt}\chi_k = \{\xi : 2^{k+1} \geq |\xi| \geq 2^k\}$. Then for $v \in \mathcal{S}(\mathbb{R}^d)$, we define the **Littlewood-Paley Projection** of v to be the following:

$$P_k(v) = \mathcal{F}^{-1}[\chi_k \mathcal{F}[v]]$$

where it is easily seen that

$$v = P_{\leq k_0}v + \sum_{k > k_0} P_k(v)$$

For sufficiently regular v, $\lim_{k_0 \to \infty} P_{\leq k_0}(v) = 0$. Also note, that $\operatorname{spt}\chi_k$ is essentially all ξ such that $|\xi| \simeq 2^k$.

Lemma 2.9 (Littlewood-Paley Characterization of $C^{0,\alpha}$). *For $v \in C^{0,\alpha}(\mathbb{R}^d)$,*

$$[v]_{C^{0,\alpha}} \simeq \sup_{k \in \mathbb{Z}}\{2^{k\alpha}||P_k v||_{L^\infty}\}$$

Proof. To show the $\gtrsim$ direction, it suffices to consider $k = 0$. In other words, we want to show that $|P_0 v| \lesssim [v]_{C^0,\alpha}$. Noting that $P_0(v) = \int \check{\chi}_0(x-y)v(y)dy$ is finite and definite since v is Schwartz, we may write

$$\int \check{\chi}_0(x-y)(v(y)-v(x))dy \leq [v]_{C^{0,\alpha}} \int \check{\chi}_0(x-y)|x-y|^\alpha dy$$

The final step of factoring out the seminorm is valid after taking a supremum inside the integral.

To see the other direction, consider

$$v(x) - v(y) = P_{\leq k_0}v(x) - P_{\leq k_0}v(y) + \sum_{k>k_0} P_k v(x) - P_k v(y)$$

Choose k_0 so that $|x-y| \simeq 2^{-k_0}$. Then, fixing k for a moment, we have

$$||P_k v||_{L^\infty} \simeq \sup_x \left\{ \| \int \chi_k(\xi) e^{i\xi x} \int v(y) e^{-i\xi y} dy d\xi \| \right\}$$

$$\int \chi_k(\xi) e^{i\xi x} \int v(y) e^{-i\xi y} dy d\xi \lesssim 2^{-k} \iint v(y) e^{i\xi(x-y)} dy d\xi \lesssim 2^{-k} \sup_{x\neq y}\{|x-y|^\alpha |v(x)-v(y)|\}$$

The final step is a geometric argument, which only holds for the special case $0 < \alpha < 1$. The rest follows from factoring the $|x-y|^\alpha$ into the 2^{-k}, and dividing again.[9]

Finally, summing over k gives

$$\sum_{k\geq k_0} ||P_k v||_{L^\infty} \lesssim \sum_{k\geq k_0} 2^{-k\alpha}[v]_{C^{0,\alpha}} \simeq |x-y|^{-\alpha}[v]_{C^{0,\alpha}}$$

Using this, we take

$$\begin{aligned} |P_{\leq k_0}v(x) - P_{\leq k_0}v(y)| &\leq ||\nabla P_{\leq k_0} v||_{L^\infty}|x-y| \\ &\leq \sum_{k\leq k_0} ||\nabla P_k v||_{L^\infty}|x-y| \\ &\lesssim \sum_{k\leq k_0} 2^{k-k\alpha}[v]_{C^{0,\alpha}} \end{aligned}$$

The rest follows from taking a supremum, noticing that the sum $k \leq k_0$ is a finite sum. □

Now, using the lemma, we are ready to attempt a proof of the constant coefficient interior regularity theorem.

L-P theory. We take

$$P(P_k u) = P_k f \mapsto_{\mathcal{F}} a_{jl}\xi^j \xi^l (\widehat{P_k u}) = \widehat{P_k f}$$

thus, by uniform ellipticity we may write that (for $\tilde{\chi}_k$ supported on a slightly wider annulus than

[9] I'm not 100% on the rest of this proof, but the details were pretty fast in lecture.

χ_k,

$$\widehat{P_k u} = \frac{1}{a^{jl}\xi_j\xi_l}\widehat{P_k f}\tilde{\chi}_k = \frac{2^{2k}}{a^{jl}\xi_j\xi_l}\widehat{P_k f}\frac{1}{2^{2k}}$$

where we call $\eta_k := \frac{2^{2k}}{a^{jl}\xi_j\xi_l}$. Taking an inverse fourier transform, $P_k u$ ends up as $\check{\eta_k} * P_k f 2^{-2k}$, and we have then that

$$||P_k u||_{L^\infty} \leq c2^{-2k}||P_k f||_{L^\infty} \lesssim ||f||_{C^{k-2,\alpha}(\mathbb{R}^d)}$$

□

We can get another proof by using a compactness argument to induce a contradiction.

Compactness + Contradiction. Assume for contradiction's sake that there exist some sequence of functions and coefficients a_n^{jk}, u_n, f_n such that $P_n u_n = f_n$, and $[u_n]_{C^{2,\alpha}} = 1$, $[f_n]_{C^{0,\alpha}} \leq \frac{1}{n}$. After translation, we can take some point $|\eta_n| = 1$, and say $|D^2 u_n(\eta_n) - D^2 u_n(0)| \geq c > 0$

Second, the second-order taylor expansion of u, with the first-order term absorbed into $\tilde{f}_n$ still satisfies

$$P_n v_n = P_n(u_n - \frac{1}{2}x^2 D^2 u_n(x)) = \tilde{f}_n$$

Thus, in the limit, $Pv = 0$, with $[D^2 v] \leq 1$ somewhere. Thus, somewhere, $D^2 v(\eta) \neq 0$ by compactness. By Liouvilles theorem, v must be a constant, which contradicts $D^2 v \neq 0$.

□

2.4 Maximum Principles

Importantly, the principles dealt with in this section *require* that our equation be Scalar. Although the results of the previous sections can be extended to nonscalar elliptic PDEs, maximum priniples in general cannot. For the duration of this discussion, we assume P is of non-divergence form, with coefficients a, b, c such that a is elliptic, and all are bounded.

$$Pu = (-a^{jk}\partial_j\partial_k + b^j\partial_j + c)u$$

The basic idea here is to generalize the concept of a convex function on $\mathbb{R}$. In 1 dimension, the principle for convex functions can be stated as: for a convex function $u : [0,1] \to \mathbb{R}$ $\max_{[0,1]} u = \max_{\partial[0,1]} u$. In as many words, convex functions either achieve their maximum value on the boundary of their domains, or they are constant.

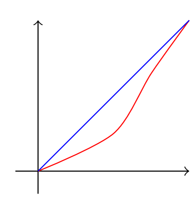


Figure 4: A Convex Function achieves it's maximum on the boundary of it's domain.

There are two appropriate generalizations of this idea to multiple dimensions. The first is to require the Hessian matrix to be positive definite. However, this ends up being too restrictive, since the existence of the Hessian requires twice-differentiability to begin with. The second is to consider functions which are subsolutions to elliptic PDEs, which is the approach considered here.

Definition 20 (Classical Subsolution). $u \in C^2(U)$ is a **classical subsolution** to the equation $Pu = 0$ if $Pu \le 0$ on U.

Theorem 2.10 (Weak Maximum Principle). *Let $u \in C^2(U) \cap C(\overline{U})$, where $U \subseteq \mathbb{R}^d$ is open, bounded and connected, and $Pu \le 0$, with $c = 0$. Then $\max_{\overline{U}} u = \max_{\partial U} u$.*

Proof. Consider a strict subsolution $Pu < 0$, and assume for the purpose of contradiction that it attains a local maximum at $x_0 \in U^o$. Then $Du(x_0) = 0$, and $D^2 u(x_0)$ has no positive eigenvalues by the second derivative test. But

$$\begin{aligned} Pu(x_0) &= -a^{jk}\partial_j\partial_k u(x_0) + \cancelto{0}{b^j\partial_j u(x_0)} \\ &= -a^{jk}\partial_j\partial_k u(x_0) \\ &= -\operatorname{tr}\big(aD^2u\big) \end{aligned}$$

Since $a > 0$ by uniform ellipticity, and $D^2 u \le 0$ by the derivative test, $-\operatorname{tr}\big(aD^2u\big) \ge 0$, so $Pu(x_0) \ge 0$. This contradicts the strict subsolution property, so we have shown that no interior local maxima exists for strict subsolutions.

Considering now $Pu \le 0$, we approximate $u_\epsilon = u + \epsilon v$, where v is a strict subsolution $Pv < 0$. Then $u_\epsilon \to u$ on U and is a strict subsolution, so the previous argument applies. □

In the preceding proof, flipping the signs gives the weak minimum principle for supersolutions by an identical proof. If u is a solution $Pu = 0$, then u is clearly both a sub and supersolution. Thus, $Pu = 0 \Rightarrow \max_{\overline{U}} |u| = \max_{\partial U} |u|$.

Corrolary (WMP for $c \ge 0$). *Under otherwise identical conditions as the Weak Maximum Principle, with $c \ge 0$, we have that*

$$\begin{cases} Pu \le 0 & \Rightarrow \quad \max_{\overline{U}} u \le \max_{\partial U} u^+ \\ Pu \ge 0 & \Rightarrow \quad \min_{\overline{U}} u \le \min_{\partial U}(-u^-) \end{cases}$$

where

$$u^+ = \begin{cases} u(x) & \forall x : u(x) \ge 0 \\ 0 & \text{otherwise} \end{cases} \qquad u^- = \begin{cases} 0 & \text{otherwise} \\ -u(x) & \forall x : u(x) \le 0 \end{cases}$$

Proof. Let $V = \{x \in U : u(x) > 0\}$. Cut V into components and individually apply the maximum principle to $Qu = Pu - cu$ on V. For Q, $\max_{\overline{V}} u \le \max_{\partial V} u$, which implies the rest. □

Theorem 2.11 (Comparison Principle). *Let U be open, bounded and connected in $\mathbb{R}^d$, with P satisfying the usual hypotheses, and $c \ge 0$. Let $u, v \in C^2(U) \cap C(\overline{U})$, with $Pu \le 0$, $Pv \ge 0$ on U, and $u \le v$ on ∂U. Then $u \le v$ on all of U.*

Proof. $u - v$ is a subsolution, so an application of the Weak Maximum Principle yields the result. □

Theorem 2.12 (Strong Maximum Principle)**.** *Let U be open, bounded and connected in $\mathbb{R}^d$, with P satisfying the usual hypotheses, and $c = 0$. Let $u \in C^2(U) \cap C(\overline{U})$, and let $Pu \leq 0$. If at any $x_0 \in U^o$ $u(x_0) = \max_{\overline{U}}$, then u is constant.*

Our proof of this theorem will make use of Hopf's Lemma, which is included below.

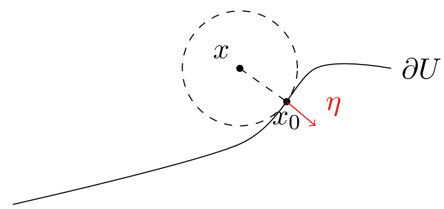


Figure 5: $\frac{\partial u}{\partial \eta}$ is increasing outward.

Lemma 2.13 (Hopf's Lemma)**.** *Let U be open, bounded and connected in $\mathbb{R}^d$. For some $x_0 \in U$,*

i. There exists $x \in U$, $0 < r \in \mathbb{R}$ such that $B_r(x) \subseteq U$, and $B_r(x) \cap \partial U = \{x_0\}$
ii. $u(x_0) \geq u(x)$ for all $x \in \overline{B_r(x)}$, and $u(x_0) > u(x)$ for all $x \in B_r(x)^o$

Then, letting η be the unit normal outward pointing vector to $B_r(x)$,

$$\left.\frac{\partial u}{\partial \eta}\right|_{x=x_0} > 0$$

Proof. Here, our goal is to compare a supersolution to a subsolution. WLOG, we set x to be the origin, and write the function $v(x) = e^{-\mu r^2} - e^{-\mu|x|^2}$, so that $v|_{\partial B_r(0)} = 0$. Moreover, $Pv \geq 0$ on $V = B_r \setminus B_{\frac{r}{2}}$. Consider the function $w = \epsilon v + u(x_0)$. Then on V, $Pw = P(\epsilon v) + Pu(x_0) = P(\epsilon v) \geq 0$ for $\mu \gg 0$. On ∂B_r, $w = u(x_0) \geq u$, and $w = \epsilon v + u(x_0)$. Therefore, $u(x_0) > u(x) - \epsilon v$ for sufficiently small ϵ, implying that $w \geq u$. By the comparison principle, $w > u$ on V, so $\frac{\partial u}{\partial \eta} \geq \frac{\partial w}{\partial \eta} > 0$, and we're done (checking that $\frac{\partial w}{\partial \eta} > 0$ is not difficult). □

Now, we can prove the strong maximum principle.

Proof of the Strong Max Principle. Consider $V = \{x \in U : u(x) < \sup_{\overline{U}} u\}$. Let $x_0 \in U$, and let $u(x_0)$ attain the maximum. Then find the largest r such that Hopf's lemma may be applied to $x \in V$, where $B_r(x) \subset V$, and $x_0 \in B_r(x_1) \cap \partial V$. Then by Hopf's lemma, $\frac{\partial u}{\partial \nu} > 0$, but this contradicts that $u(x_0)$ is a local maximum. Thus, V must be empty, i.e. $u(x)$ is constant. □

I missed a lecture on March 11. The following theorem is stated without proof, and a proof can be found in Lerner's Carleman's Estimates.

Theorem 2.14 (Aronszajn's theorem)**.** *Let P be an elliptic operator*

$$Pu = -\partial_j(a^{jk}\partial_k u) + b^j \partial_j u + cu$$

If $Pu = 0$ in U, and $u = 0$ on an open subset of $W \subseteq U$, then $u \equiv 0$ on U.

3 Linear Hyperbolic PDEs

The precise meaning of Hyperbolicity varies a bit, and arguably only precise at all in the constant coefficient case. However, there are a few guidelines and heuristics that can be used to establish a working definition.

- Hyperbolic PDEs are typically *evolutionary* equations, in that they have a contextually clear time variable. We will notate this by settings $x_0 = t$, and writing $\mathbb{R}^{1+d}$.
- The number of t-derivatives is the same as the number of ∇-derivatives (spatial).
  - e.g. the classical wave equation $(\partial_t^2 - \Delta)\phi = 0$.
  - e.g. any transport equation $(\partial_t + X^j\partial_j)\phi = 0$.
  - *not* e.g. the heat equation $(-\partial_t + \Delta)u = 0$.
  - *not* e.g. the Schrödinger equation $(i\partial_t + \Delta)u = 0$.
- Well-Posedness of the initial value problem (where N is the order of ∂_t):

$$\begin{cases} P\phi = 0 \\ (\phi, \partial_t\phi, \dots, \partial_t^{N-1}\phi) = (g_0, g_1, \dots g_{N-1}) \end{cases}$$

In the context of this class, it will usually be best to find algebraic conditions guaranteeing the well-posedness condition. There is also a close relationship between hyperbolicity and the existence of energy estimates.

Example (Linear, Constant Coefficient Systems). Let $\Phi : \mathbb{R}^{1+d} \to \mathbb{R}^n$, with $\mathbf{A}$ a an $n \times n$ matrix of $d \times d$ matrices.[10]

$$\partial_t\Phi + A^j\partial_{x^j}\Phi = F$$

How do we guarantee uniqueness of the initial value problem? One approach would be to use an energy estimate to place algebraic conditions on $\mathbf{A}$.[11] Here, we use the notation that indices in parentheses (e.g. $\phi^{(k)}$) range over $1, \dots, n$, while indices not in parentheses range over $1, \dots, d$.

$$\begin{aligned} \int_{\mathbb{R}^d} \phi^{(k)} F_{(k)} dx &= \int_{\mathbb{R}^d} \phi^{(k)}\partial_t\phi_{(k)} + \phi^{(k)}(A^j\partial_{x^j})^{(k)}\phi_{(k)} dx \\ &= \frac{1}{2}\left(\int_{\mathbb{R}^d} |\Phi|^2 dx + \int_{\mathbb{R}^d} (A^j)^{(k)}_{(\ell)}\phi^{(k)}\partial_{x^j}\phi^{(\ell)} dx - \int_{\mathbb{R}^d} (A^j)^{(k)}_{(\ell)}(\partial_{x^j}\phi^{(k)})\phi^{(\ell)} dx\right) \\ &= \frac{1}{2}\left(\int_{\mathbb{R}^d} |\Phi|^2 dx \left((A^j)^{(k)}_{(\ell)} - (A^j)^{(\ell)}_{(k)}\right)\phi^{(k)}\partial_j\phi^{(\ell)}\right) \end{aligned}$$

We can see from this final integral term that when $(A^j)^{(k)}_{(\ell)}$ is symmetric, we get a nice energy estimate of the form $\frac{1}{2}||\Phi||^2_{L^2_x}$. This is the type of algebraic constraint yielding energy estimates we seek.

[10] There's definitely a more elegant understanding of this, but this is sufficient for our use case.

[11] I think professor Oh may have implicitly assumed that A^j is a diagonal matrix, and there really should be a $A^j_k\partial_{x^k}$ in here somewhere.

As an aside, the trick we used in the integration above to obtain our symmetry result was simply to split the second term in half, and integrate only half of it by parts.

Theorem 3.1. *If A is a constant coefficient operator*

$$\partial_t \Phi + A^j \partial_{x^j} \Phi = F$$

is hyperbolic (i.e. the initial value problem is well-posed in L^2) if and only if $\mathbf{A}$ is symmetric.

The details of this theorem will be explored later, but it follows essentially from using the energy estimate to show uniqueness.

Finding a non-hyperbolic counterexample is an excercise in Fourier analysis, since non-symmetry produces plane wave solutions.

Example (First order Wave Equation). We can make $\Box\phi = f$ into a first order equation by converting to the system $\phi = \partial_t \phi$, so we have $\partial_t \psi = \Delta\phi - f$. Then

$$\partial_t \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} - \begin{bmatrix} 0 \\ f \end{bmatrix}$$

After a fourier transform and subsequent diagonalization

$$\widehat{\partial_t \begin{bmatrix} \phi \\ \psi \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ -|\xi|^2 & 0 \end{bmatrix} \widehat{\begin{bmatrix} \phi \\ \psi \end{bmatrix}} - \widehat{\begin{bmatrix} 0 \\ f \end{bmatrix}} = \begin{bmatrix} i|\xi| & 0 \\ 0 & -i|\xi| \end{bmatrix} \widehat{\begin{bmatrix} \phi \\ \psi \end{bmatrix}} - \widehat{\begin{bmatrix} 0 \\ f \end{bmatrix}}$$

The energy estimate holds in the diagonalized variables, but is no longer unique.

3.1 Variable Coefficient Wave Equations

The reference for this section is Hans Ringström's *The Cauchy Problem in General Relativity*, Chapters 6 and 7. Our objective is to derive energy estimates for

$$\begin{cases} P\phi = f & \mathbb{R}_+ \times \mathbb{R}^d \\ (\phi, \dot{\phi}) = (g, h) & \{t = 0\} \times \mathbb{R}^d \end{cases} \tag{7}$$

A preliminary and important tool we will use frequently throughout this section is Grönwall's inequality.

Theorem 3.2 (Grönwall's Inequality)**.** *Suppose $E(t) \in C_t([0,T])$, with $E \geq 0$, $r(t) \in L^1_t([0,T])$, $r \geq 0$, and*

$$E \leq E(0) + \int_0^t r(t')E(t')dt'$$

Then

$$E(t) \leq E(0) \exp\left(\int_0^t r(t')dt'\right)$$

Proof. This method of proof is known as the *Bootstrap* method. Essentially, the argument goes by plugging the right-hand side of the result into the hypothesis, and deriving a stronger result which in turn proves the assumption. Let t be some time $0 \le t < T$.

$$\begin{aligned} E(t) &\le E_0 + \int_0^t r(\tau)E(\tau)d\tau \\ &\le E_0 + E_0 \int_0^t r(\tau) \exp\left(\int_0^\tau r(t')dt'\right) d\tau \end{aligned}$$

Now, we make a change of variables $dR(\tau) = r(\tau)d\tau$. Directly integrating the previous expression yields

$$E(t) \le E_0 + E_0 \left(\exp\left(\int_0^t r(t')dt'\right) - 1\right) = E_0 \exp\left(\int_0^t r(t')dt'\right)$$

If instead, before the change of variables we fix $\tau < t \le T$, we have essentially completed the entire proof. □

We now introduce some notation that will be used in describing variable coefficient wave equations. First, we set the operator

$$P\phi = \partial_\mu(g^{\mu\nu}\partial_\nu\phi) + b^\nu\partial_\nu\phi + c\phi$$

Definition 21. The metric $g^{\mu\nu}$ is called **lorentzian** if it is symmetric, and has signature $(-1, 1, \ldots, 1)$. For our description of wave equations, g is assumed lorentzian.

Definition 22. We use the **Einstein Summation Convention**. Greek indices run over $\{0, \ldots, d\}$ (i.e. all variables), and latin indices run over $\{1, \ldots, d\}$ (i.e. spatial variables). Repeated indices imply summation, et cetera.

The overall idea to obtain energy inequalities is to multiply by $\partial_t\phi$, and integrate by parts. The algebra here is a bit involved, so we treat only the second-order terms.

$$\begin{aligned} \partial_\mu(g^{\mu\nu}\partial_\nu\phi)\partial_0\phi &= (-\partial_0^2\phi)\partial_0\phi + \partial_j(g^{jk}\partial_k\phi)\partial_0\phi \\ &= \partial_0\left(-\frac{1}{2}(\partial_0\phi^2)\right) + \partial_j\left(g^{jk}\partial_k\phi\partial_0\phi\right) - (g^{jk}\partial_k\phi)\partial_j\partial_t\phi \\ &= -\frac{1}{2}\partial_0\left((\partial_0\phi)^2 + g^{jk}\partial_j\phi\partial_k\phi\right) + \partial_j(g^{jk}\partial_k\phi\partial_0\phi) + \frac{1}{2}(\partial_0 g^{jk})\partial_j\phi\partial_k\phi \end{aligned}$$

Now, we integrate by parts over the region $\mathcal{R}_{t_0}^{t_1} = (t_0, t_1) \times \mathbb{R}^d$, assuming a vanising boundary term.

We denote the **Cauchy hypersurface** $\Sigma_t = \{\{t\} \times \mathbb{R}^d\}$.

$$\int_{\mathcal{R}_{t_0}^{t_1}} \partial_\mu (g^{\mu\nu} \partial_\nu \phi) \partial_0 \phi dx dt$$

$$= - \int_{\Sigma_{t_1}} \frac{1}{2} \left((\partial_0 \phi)^2 + g^{jk} \partial_j \phi \partial_k \phi \right) dx + \int_{\Sigma_{t_0}} \frac{1}{2} \left((\partial_0 \phi)^2 + g^{jk} \partial_j \phi \partial_k \phi \right) dx$$

$$+ \cancelto{0}{\lim_{r \to \infty} \int_0^t \int_{\partial B_r} \nu_j (g^{jk} \partial_k \phi \partial_0 \phi) dA dt}$$

This is our desired conservation law! At times t_1 and t_0, $||D\phi||_{L^2}$ is the same (contracted w/ the metric)! We now use this to prove the following lemma.

Lemma 3.3. *For $\phi \in C_t H^1$*

$$\sup_{t \in [0,T]} ||\vec{\phi}||_{H^1} \leq C_T \left(||\phi(t=0)||_{H^1} + \int_0^t ||P\phi||_{L^2} dt \right)$$

Proof. Using the energy law we derived above, define the energy of the solution ϕ to be a function of time

$$E[\phi](t) = \frac{1}{2} \int_{\Sigma_t} (\partial_t \phi)^2 + \partial_j \phi g^{jk} \partial_k \phi dx$$

Then,

$$\begin{aligned} E[\phi](t_1) &= E[\phi](0) - \iint_{\mathcal{R}_0^{t_1}} \partial_\mu (g^{\mu\nu} \partial_\nu \phi) \partial_t \phi dx dt + \frac{1}{2} \iint_{\mathcal{R}_0^{t_1}} (\partial_t g^{jk}) \partial_j \phi \partial_k \phi dx dt \\ &= E[\phi](0) + \iint_{\mathcal{R}_0^{t_1}} (P\phi) \partial_t \phi dx dt + \iint_{\mathcal{R}_0^{t_1}} \partial_t \phi (b^\mu \partial_\mu \phi + c\phi + (\partial_t g^{jk}) \partial_j \phi \partial_k \phi) dx dt \end{aligned}$$

The final term represents an error of sorts, so we define it thusly:

$$\mathcal{E}_0^{t_1} = \iint_{\mathcal{R}_0^{t_1}} |\partial_t \phi (b^\mu \partial_\mu \phi + c\phi + (\partial_t g^{jk}) \partial_j \phi \partial_k \phi)| dx dt$$

Then, we have from Grönwall's inequality that

$$\sup_{t \in [0,T]} E[\phi](t) \leq E[\phi](0) + \sup_{t \in [0,T]} \left| \iint_{\mathcal{R}_0^T} (P\phi) \partial_t \phi dx dt \right| + \mathcal{E}_0^T$$

Note also that

$$\int_{\mathbb{R}^d} |\phi(t)|^2 dx = 2 \int_0^t \int_{\mathbb{R}^d} \partial_t \phi \phi dx dt \leq 2 \int_0 |E(t')|^{1/2} \int |\phi|^2 dx dt' + \int |\phi|^2 (0) dx$$

□

A Frequently Cited Theorems and Definitions

A.1 Real Analysis

Theorem A.1 (Existence of Smooth Partitions of Unity).

A.2 Functional Analysis

Definition 23. Let X be a real vector space. A map $p : X \to \mathbb{R}$ is called a **sublinear functional** if it satisfies the following for all $x, y \in X$:

1. $p(x + y) \leq p(x) + p(y)$,
2. For all $\lambda \geq 0$, $p(\lambda x) = \lambda p(x)$.

Theorem A.2 (Hahn-Banach). *Let X be a real vector space, p a sublinear functional on X, M a subspace of X, and f a linear functional on M such that $f(x) \leq p(x)$ for all $x \in M$. Then there exists a linear functional F on X such that $F(x) \leq p(x)$ for all $x \in X$, and $F_{|M} = f$.*

Proof. The following proof is due to Folland. We first show that for $x \in X \setminus M$, f may be extended to a linear functional g on $M + \mathbb{R}x$ which satisfies $g(y) \leq p(y)$. For $y_1, y_2 \in M$, we have

$$f(y_1) + f(y_2) = f(y_1 + y_2) \leq p(y_1 + y_2) \leq p(y_1 - x) + p(x + y_2)$$

Rearranging gives

$$f(y_1) - p(y_1 - x) \leq p(x + y_2) - f(y_2)$$

Since this applies to every $y_1, y_2 \in M$, we have

$$\sup_{y \in M} \{f(y) - p(y - x)\} \leq \inf_{y \in M} \{p(x + y) - f(y)\}$$

Let α be any number which satisfies

$$\sup_{y \in M} \{f(y) - p(y - x)\} \leq \alpha \leq \inf_{y \in M} \{p(x + y) - f(y)\}$$

and define $g : M + \mathbb{R}x \to \mathbb{R}$ by $g(y + \lambda x) = f(y) + \lambda\alpha$. g is linear by the linearity of f and multiplication by λ. Furthermore, $g_{|M} = f$, since any input in M has $\lambda = 0$, which gives $g(y) \leq p(y)$ for $y \in M$. Moreover, if $\lambda > 0$, and $y \in M$, we have

$$g(y + \lambda x) = \lambda \left[f\left(\frac{y}{\lambda}\right) + \alpha \right] \leq \lambda \left[f\left(\frac{y}{\lambda}\right) + p\left(x + \frac{y}{\lambda}\right) - f\left(\frac{y}{\lambda}\right) \right] = p(y + \lambda x)$$

If instead, we say $\lambda = -\mu < 0$,

$$g(y + \lambda x) = \mu \left[f\left(\frac{y}{\mu}\right) - \alpha \right] \leq \mu \left[f\left(\frac{y}{\mu}\right) - p\left(-x + \frac{y}{\mu}\right) - f\left(\frac{y}{\mu}\right) \right] = p(y + \lambda x)$$

So, we have $g(z) \leq p(z)$ for all $z \in M + \mathbb{R}x$.

Importantly, the above logic doesn't really depend on the fact $x \in X \setminus M$. If F is any linear extension of f, then $F \leq p$ on it's domain, which shows that the domain of a maximal linear functional satisfying $F \leq p$ must be X. The family $\mathcal{F}$ of linear extensions F of f satisfying $F \leq p$ is partially ordered by inclusion when we consider maps from subspaces of X to $\mathbb{R}$ as subsets of $X \times \mathbb{R}$. Since the union of any increasing family of subspaces of X is also a subspace of X, the union of a linearly ordered subfamily of $\mathcal{F}$ also lies in $\mathcal{F}$. So, we may invoke Zorn's lemma to guarantee the existence of a maximal element $F \in \mathcal{F}$, which completes the proof. □

Theorem A.3 (Open Mapping). *Let X, Y be Banach spaces. If $T \in L(X, Y)$ is surjective, then T maps open sets to open sets.*

Proof. See Folland 5.10. □

A.3 L^p Spaces

Theorem A.4 (Hölder's Inequality).

Theorem A.5 (Minkowski Inequality).

B References