Real Analysis

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Class: Math 104
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Contents

Introduction

OH: Thursdays from 1:00-3:30.
Grading: 20% HW, 10% attendance. 30% Midterm, 40% Final Exam. Homework due every tuesday in class.

1 $\mathbb{Q}$ and $\mathbb{R}$

Study of class is sequences, series and functions, convergence, continuity, differentiability and integrability. In order to do this, we need an idea of numbers.
In order to do this, we extend $\mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C}$.

1.1 Ring Theory, Rev

$$\begin{aligned}
\mathbb{N}, \mathbb{Z}^+, \mathbb{Z}^* : &\qquad 1, 2, 3, ... \\
\mathbb{Z}^{\geq 0} : &\qquad 0, 1, 2, 3... \\
\mathbb{Z} : &\qquad ... - 5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5...
\end{aligned}$$

Integers are an integral domain. We want the field extension of $\mathbb{Z}$, so we create the multiplicative inverse of every object.

$$\mathbb{Q} : \qquad \frac{m}{n} \Big| \{m, n \in \mathbb{Z}, n > 0, \gcd(m, n) = 1\}$$

Definition 1.1 (Field)**.** Every field $(\mathbb{F}, +, \times)$ must satisfy the following three conditions:

- $(\mathbb{F}, +)$ is an abelian group with identity 0.
  - $+$ is a binary operator on $\mathbb{F}$, $\mathbb{F}$ is closed under $+$.
  - $+$ is associative, e.g. for $a, b, c \in \mathbb{F}$, we have $(a + b) + c = a + (b + c)$.
  - $\exists e \in \mathbb{F}$ such that $\forall f \in \mathbb{F} e + f = f + e = e$.
  - Every element is invertible. $\forall f \in \mathbb{F} \exists (-f) : f + -f = e_+$.
  - $+$ commutes. for $a, b \in \mathbb{F}$, we have $a + b = b + a$.
- $(\mathbb{F}, +, \times)$ is a commutative, unital ring.
  - $\mathbb{F}$ is closed under $\times$.
  - $(\mathbb{F}, \times)$ associates.
  - $\exists e_\times$ such that for every $f \in \mathbb{F}, e_\times \times f = f \times e_\times = f$.
  - $\times$ commutes.
  - $(+, \times)$ distribute. For $a, b, c \in \mathbb{F}$, $(a + b) \times c = (a \times c) + (b \times c)$.

notice that in $\mathbb{Q}$, there is no inverse for 0. Any nonzero element, however, has a multiplicative inverse.

Statement 1.1 ($(\mathbb{Q}, <)$ is ordered.)**.**

$$a < b \Leftrightarrow a + (-b) < 0$$

1.2 Ordered Sets/Upper Bounds

Definition 1.2 (Ordered Set)**.** $(X, \prec)$ is an ordered set if the following hold.

- for any two $x, y \in X$, one and only one of the following holds: $\begin{cases} x \prec y \\ y \prec x \\ y = x \end{cases}$.
- For any $x \prec y$, $y \prec z$ we have $x \prec z$.

Example 1.2.1 (Power Set)**.** We want to know if $(2^x, \subseteq)$ is ordered. Let's consider the examples $A = \{1, 2\}, B = \{-1, 0\} \in 2^{\mathbb{Z}}$. Partially ordered.

Lemma 1.0.1. *If $(X, \prec)$ is ordered, we can conclude that any $S \subseteq X \Rightarrow (S, \prec)$ is ordered.*

The preceding lemma gives us some notion of upper and lower bounds for a subset $S \subseteq S$.

Definition 1.3 (Upper Bound)**.** The upper bound for a subset $S \subseteq X$ is $x \in X$ such that $\forall a \in S \begin{cases} a \prec x \\ a = x \end{cases}$.

Example 1.2.2.

- $X = \mathbb{Q}$, $S = \mathbb{Q}$, then S has no upper bound.
- $S = \{x \in \mathbb{Z} | x^2 \leq 2\} \subseteq \mathbb{Z}$. 1 is an upper bound.

Definition 1.4 (Least Upper Bound)**.** For $S \subseteq X$ with $(X, <)$ ordered, an element $x_0 \in X$ is the least upper bound of S if

- x_0 is an upper bound of S in X.
- Any upper bound x of S satisfies $x \geq x_0$.

We call the least upper bound $\sup_X S$.

Lemma 1.0.2 (Least Upper Bound is Unique)**.** *The least upper bound, when it exists, is unique.*

Proof. Take some x_1, x_2 as least upper bounds of S. Then x_1, x_2 both upper bounds of S. Since, we the definition of upper bound, and both are the least, $x_1 \prec x_2$, and $x_2 \prec x_1$, so $x_1 = x_2$. $\square$

Consider the question of whether, with some set of upper bounds, *must* a least upper bound exist?
Consider the following

Example 1.2.3. $X = \mathbb{Q}$, with $S = \{x \in \mathbb{Q} | x^2 \leq 2\} \subseteq \mathbb{Q}\}$.

I think this is going to introduce us to the concept of limiting polynomials until we find weird solutions like $\sqrt{2}$.
Continuing the previous example, we have a bunch of upper bounds, e.g. 2, and every rational larger than 2 is an upper bound.
We can check that 1.5 is an upper bound is an upper bound.
If we take the set of upper bounds, we draw it like

Bassically, we already know how to prove $\sqrt{2}$ isn't rational, so we want to take some

$$\left.\begin{array}{rl} x = x_0 - \frac{x_0^2 - 2}{x_0 + 2} \in \mathbb{Q} & x > x_0 \\ x^2 = 2 + \frac{2(x_0^2 - 2)}{(x_0 + 2)^2} < 2 & x \in S \end{array}\right\} \Rightarrow x_0 \text{ is not an upper bound, since this is a contradiction}$$

Now, we want to find $x < x_0$ such that x is an upper bound.

Definition 1.5 (Least Upper Bound Property)**.** Say that an ordered set $(X, \prec)$ has the least upper bound property if any $S \subseteq X$ with some upper bound has the least upper bound.

[1]

1.3 Constructing the Reals

We want some field $\mathbb{R}$ so that we preserve all the operations of $\mathbb{Q}$ as a field. We also want these to be compatible with each other, e.g. equivalence relations, ordering remain the same.

Definition 1.6 (Ordered Field)**.** Let X be a set with $\left.\begin{array}{cc} (X, +, \times) & \text{field} \\ (X, \prec) & \text{ordered set} \end{array}\right\}$ is an ordered field if

1. $x \prec y$, $z \in X \Rightarrow x + z \prec y + z$.
2. $x > 0, y > 0 \Rightarrow x \times y > 0$.

Maybe we consider taking the power set of $\mathbb{Q}$, and choose special elements from the power set to get the reals.
Consider $2^{\mathbb{Q}}$, the power set of $\mathbb{Q}$, the set of all subsets of $\mathbb{Q}$.
How do we realize $\mathbb{Q} \subseteq 2^{\mathbb{Q}}$? Let's make an injective map from $\mathbb{Q} \to 2^{\mathbb{Q}}$

$$\begin{array}{rc} \varphi : & \mathbb{Q} \to 2^{\mathbb{Q}} \\ & \mathbb{Q} \cong \varphi(\mathbb{Q}) \subseteq 2^{\mathbb{Q}} \end{array}$$

Since we force φ injective, we want it to fulfill the property that

$$\varphi(q_1) = \varphi(q_2) \Rightarrow \{x \in \mathbb{Q} | x < q_1\} = \{x \in \mathbb{Q} | x < q_2\}$$

We can get an inverse map for φ, namely the supremum. We can make a dedekind cut

[1] TODO: does it mean to be ordered if only one $S \subseteq X$ has the LUB, or if all subsets have it. Doesn't that definition reference itself in its own construction?

Definition 1.7 (Cut)**.** A cut is defined by

1. $C \subseteq \mathbb{Q} C \neq \emptyset, C \neq \mathbb{Q}$
2. if $c \in C$, then for any $x < c$, $x \in C$.
3. If $c \in C$, then we can find some $x \in \genfrac{}{}{0pt}{}{(-\infty, q)}{C}$ so that $x > c$.

We already checked that every element in $\varphi(\mathbb{Q})$ is a cut, but not every cut is in $\varphi(\mathbb{Q})$.
One set that is a cut, but does not live in the image of φ is $\left\{ x \in \mathbb{Q} | \genfrac{}{}{0pt}{}{x^2 < 2}{\text{or} x < 0} \right\}$.
Dedekind then allows us to define the reals as the set of all cuts. Let $R =$ the set of all cuts. We need to add on addition and multiplication.

$$\begin{array}{ccc} q_1 & \mapsto & (-\infty, q_1) \\ + & & + \\ q_2 & \mapsto & (-\infty, q_2) \\ q_1 + q_2 & \mapsto & (-\infty, q_1 + q_2) \end{array}$$

Same goes for multiplication. We can demonstrate that this becomes an ordered field. Check the appendix in rudin.

Theorem 1.1 (Dedekind)**.** $(R, +, \times, \subset)$ *is an ordered field as an extension of* $(\mathbb{Q}, +, \times, <)$.

Example 1.3.1 (Writing down $\sqrt{2}$)**.** This is as simple as taking the interval $\{x \in \mathbb{Q}|^2 < 2 \text{ or } x < 0\}$.[2]

Definition 1.8 (Maximum, Minimum)**.** Consider $(X, <)$, $S \subseteq X$. Then

$$\begin{array}{cc} \sup_x S & \inf_x S \\ \max_x S & \min_x S \end{array}$$

For the maximum, minimum functions, we add the additional requirement that $\max S \in S$, and that $\min S \in S$. There is no such requirement for the supremum, infimum.

Recall our definition of $\mathbb{R}$ as the set of cuts in $\mathbb{Q}$, such that a cut satisfies various properties.[3]
It also has the least upper bound property.

Theorem 1.2 (Archimedian Property of $\mathbb{R}$.)**.** *For any* $x \in \mathbb{R}^+, y \in \mathbb{R}$, *then* $\exists n \in \mathbb{Z}^+$ *so that* $n \cdot x > y$.[4]

Lemma 1.2.1 (Boundedness of $\mathbb{R}$)**.** *For any* $y \in \mathbb{R}$, *we can find* $n \in \mathbb{Z}^+$ *so that* $n \cdot 1 = n > y$.

Proof. Assume that there is no such n such that $n > y$. This directly implies that $n \leq y \forall n \in \mathbb{Z}^+$. This implies that y is an upper bound on $\mathbb{Z}^+$, and, by the LUB property, $\mathbb{Z}^+$ must have least upper bound x_0.
Consider the element $x_0 - 1 < x_0$, and assume it to be the upper bound of $\mathbb{Z}$. $x_0 - 1$ is not an upper bound of $\mathbb{Z}^+$, so we can find $N \in \mathbb{Z}^+$ so that $x_0 < x_0 + 1$, this contradicts the assumption that x_0 is an upper bound of $\mathbb{Z}^+$. Thus, $\mathbb{Z}$ cannot have a least upper bound, which gives the above paragrpah is a contradiction. $\square$

This allows us to prove the general case.

Proof. $x \in \mathbb{R}^+$, $y \in \mathbb{R}$. Consider $\frac{y}{x} \in \mathbb{R}$, since $\mathbb{R}$ is a field. Now, we apply the proven Lemma 1.2.1 to say that $\exists n \in \mathbb{Z}^+$ so that $n > \frac{y}{x}$. Since $\mathbb{R}$ is an ordered field, we're allowed to take the following: $n \cdot x > y$. But we now have the archimedian property as stated above, so we're done. $\square$

Theorem 1.3 ($\mathbb{Q}$ is dense in $\mathbb{R}$)**.** *For any* $a, b \in \mathbb{R}$ *such that* $a < b$, *we can find some* $x \in \mathbb{Q}$ *such that* $a < x < b$.

Proof. Take $n, m \in \mathbb{Z}$, with $n > 0$. Now we have the statement $a < \frac{m}{n} < b \to an < m < bn$. We take $bn - an = (b - a)n > 0$, and by the archimedian property, we found some positive integer such that $b - a > 1$, so we're done. $\square$

Lemma 1.3.1. *For* $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, *and* $\beta - \alpha > 1$, *then there exists some* $m \in \mathbb{Z}$ *such that* $\alpha < m < \beta$.

[2] I wonder how you'd write down pi?
[3] We need to prove (see Rudin) that this is indeed an ordered field.
[4] e.g. the cut of x, $C_x \supsetneq (-\infty, 0)$.

1.4 The Root Operation

$\mathbb{R}$ is closed under taking roots. Also, the definition of the root operation is unique. We are going to need to prove these things.

1.5 The Reals as an Ordered Field

Definition 1.9 ($\pm\infty$)**.**
- $\sup S = \infty$ if S has no upper bound
- $\sup S = -\infty$ if S has no lower bound

It is hard to extend addition, multiplication to $\mathbb{R} \cup \{+\infty, -\infty\}$.

We can extend $\mathbb{R}$ to larger structures by taking

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}\}$$

Gives the x, y plane. We have operations of addition and scalar multiplication, as defined per usual in a vector space.

2 Basic Topology

We have countable, uncountable sets. Countable sets, finite sets are both called *at most countable* sets.
Finite sets have a bijective map between a finite set of $\mathbb{Z}^+$ and the set.

Definition 2.1 (Countable Sets)**.** A set X is called countable if there exists a bijection from $\mathbb{Z}$ to X.

Last time, we showed that $\mathbb{Q}$ is countable using two lemmas.

Lemma 2.0.1 (Direct Products of Countable Sets are Countable)**.** *X, Y countable sets implies that $X \times Y = \{(x, y) | x \in X, y \in Y\}$ is also countable.*

Lemma 2.0.2 (Subsets of Countable Sets are at Most Countable)**.** *X countable $\Rightarrow$ any $S \subseteq X$ is at most countable.*

2.1 The Reals are Not Countable

We want to demonstrate that $\mathbb{R}$ is uncountable. We should consider the direct product of an infinite number of countable sets, and find out whether or not it is countable.

Theorem 2.1 ($\mathbb{R}$ is uncountable)**.** *There is no bijective function from $\mathbb{Z}^+ \to \mathbb{R}$.*

We've already shown that every $r \in \mathbb{R}$ has a decimal representation, so we can realize every decimal representation as the cartesian product

$$\mathbb{Z} \times \bigotimes_{i \in \mathbb{Z}} \mathbb{Z}^{\geq 0}$$

The even simpler version is that we are taking

$$\pm 1 \times \bigotimes_{i \in \mathbb{Z}} \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Even easier, we can consider the set of sequences consisting of only binary digits, e.g. $\bigotimes \{0, 1\}$.
Our question now becomes whether we are able to find a bijective function from $\mathbb{Z}^+ \to B$. We want to show that we cannot.

Lemma 2.1.1 (B is uncountable)**.** *There is no bijective function from $\mathbb{Z}^+$ to B.*

Proof. Assume that there exists some $f : \mathbb{Z}^+ \to B$ such that f is bijective. Then, we should have that $f(n) = (x_1^n, x_2^n, x_3^n \dots)$. We introduce the function

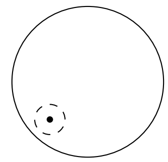
$$\begin{aligned} {}^{-} : \quad & B \to B \\ & x \mapsto \begin{cases} 0 & x = 1 \\ 1 & x = 0 \end{cases} \end{aligned}$$

From every number on the diagonal, we construct a new element $\bar{s} = (\bar{x_1^1}, \bar{x_2^2}, \dots)$. Then $\bar{s}$ is not in the image of f. If $\bar{s} = f(n)$ for some n, then $\bar{x_n^n} = x_n^n$, which is a contradiction. $\square$

We can easily prove Theorem 2.1 by noticing that by Lemmas 2.0.2 and 2.1.1, $\mathbb{R}$ cannot be countable, so we're done $\square$.
The thing is, uncountable sets are weird. We want to come up with nicer, countable ways to think about uncountable sets. Sequences of representation are one good way. A better way is a compact set, which she will explain later I guess?

2.2 Metric Spaces and Topology

Topology is basically a rigorous way of defining the "neighborhood" of a point.



We start by defining the absolute value function

Definition 2.2 (Absolute Value)**.**

$$|x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$$

We introduce the concept of a distance function

Definition 2.3 (Distance Function)**.** In general if X is a set, a function $d : X \times X \to \mathbb{R}$ is called a distance function if it satisfies the following properties

1. $d \geq 0$ and $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$.
3. It satisfies the triangle inequality: for any 3 points $x_1, x_2, x_3 \in X$, $d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)$.

Definition 2.4 (Metric Space)**.** A set X equipped with such a distance function d yields the object (X, d) which is called a metric space.

Example 2.2.1. $(\mathbb{R}, d)$ with $d(x, y) = |x - y|$ is a metric space.

Example 2.2.2. 2-d euclidean space is just the metric space $(\mathbb{R}^2, d)$ where d is defined by

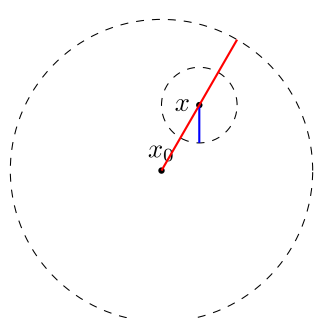
$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

We can also represent the complex numbers as such a metric space, with

$$d(z_1, z_2) = \sqrt{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}$$

Definition 2.5. A subset S of X is called an open set if any $x_0 \in S$, there exists $r > 0$ so that $B_r(x_0) \subseteq S$.

She lists a fact, that $B_r(x_0)$ is open. I kind of forget what B_r means because I zoned out here, so see the footnote for a todo![5]
Ok, so let,s take some arbitrary $x \in B_r(x_0)$. This means that $d(x, x_0) < r$, so we want $r' > 0$, so that $B_{r'}(x) \subseteq B_r(x_0)$ This might look like



where the red is r, blue is r'.
Both the empty set, X are open. Intersection of open sets are open. Union of arbitrarily many open sets are open. We will prove these. There are six minutes left in the lecture, so idk how much we're going to get through today.
Ok, we have S_1, S_2 open, take $S_1 \cap S_2 \ni x_0.$, gives $x_0 \in S_1, x_0 \in S_2$.
Recall from last time, for X, d a metric space, we have open sets

[5]TODO: what does B_r mean again??

1. $\emptyset, X$ are both open
2. The intersection of two open sets S_1, S_2 is also open.
3. S_α, $\forall \alpha \in \Lambda$ are open impies their union is also open.

We didn't prove the third one, let's do it now.

Proof. $x \in \bigcup_{\alpha\in\Lambda} S_\alpha \Rightarrow \exists \alpha \in \Lambda$ so that $x \in S_\alpha$. Because S_α is open, there is some $r > 0$ so that $B_r(x) \subseteq S_\alpha \subseteq \bigcup_{\alpha\in\Lambda}$. This proves the union is open. ☐

Property 2 only works for finitely many sets. We can take limiting sets and get them down to a single point (e.g. $\bigcap_{\alpha\in\mathbb{Z}^+}(-\frac{1}{n}, \frac{1}{n}) \to \{0\}$). We can prove this.

Proof. Let $I_n = (-\frac{1}{n}, \frac{1}{n})$, $n \in \mathbb{Z}^+$. We want some $x \neq 0 \in \bigcap_{n=1}^{\infty} I_n$, so assume $|x| > 0, n \in \mathbb{Z}^+$, we can assume some $\frac{1}{n} < |x|$, and use the archimedian property of $\mathbb{R}$ to show a contradiction. ☐

This comes equipped with an idea of closed sets

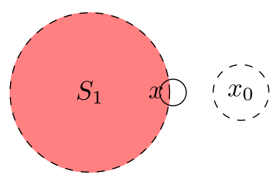
Definition 2.6 (Closed Set)**.** A subset $S \subseteq X$ is called closed if $X \setminus S$ is open.

Clsoed sets have the following properties

1. $\emptyset, X$ are closed
2. S_1, S_2 closed implies that $S_1 \cup S_2$ closed.
3. The intersection of closed sets are closed (only for finitely many).

We can just try taking the union of $\bar{I}_n$, with $\bar{I}_n = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$.

Definition 2.7 (Limit Point)**.** Let $S \subseteq X$. Then an element $x \in X$ is called a limit point of the subset S if any $B_r(x), r > 0$ contains some point in S and different from x.



x is a limit point of S_1, but x_0 is not.

Definition 2.8 (Isolated Point)**.** An isolated point is a point in S but not in S'.

Definition 2.9 (Closure)**.** The closure of a set S is the union with it's limit. $\bar{S} = S \cup S'$.

Proposition 2.1 (Closure of any set is closed)**.** *$\bar{S}$ is closed.*

Proof. It's sufficient to show that the complement is open. Let's take $\bar{S}^c$, and find $B_r(x) \subseteq \bar{S}^c$ with $r > 0$. If tehre is no such $r > 0$, then for any $r > 0$, $B_r(x) \not\subseteq \bar{S}^c$, so $\exists y \in B_r(x) \cap \bar{S}$. Since $\bar{S} = S \cup S'$, we have $y \in S$ or $y \in S'$.
If $y \in S$, we're done.
If $y \notin S$, then $y \in S'$, then $\exists z \in B_r(x) \cap S$. We just walk along y, z recursively, which givea a contradiction. I need to understand this better.:w ☐

This gives a nice way of thinking about what it means to be "dense". $\mathbb{Q}$ is dense in $\mathbb{R}$ just means that $\bar{\mathbb{Q}} = \mathbb{R}$.

Proposition 2.2. *$\bar{S}$ is the smallest closed set that contains S.*

Proof. We can take $S_\alpha \supseteq S$, with S_α closed. Then $\bar{S} = \bigcap_{S_\alpha} S_\alpha$, $\bar{S} = S_\alpha$, so $\bar{S} \supseteq S_\alpha$. The rest is left as HW. ☐

Definition 2.10 (Sequence)**.** A sequence is a set defined by a map $f : \mathbb{Z}^+ \to X$, where

$$f: \quad \begin{aligned} \mathbb{Z}^+ &\to X \\ n &\mapsto x_n \end{aligned}$$

Definition 2.11 (Limit)**.** Let x_n a sequence in X, A point $x \in X$ is called a limit of x_n if

$$\lim_{n \to \infty} d(x_n, x) = 0$$

For any $\varepsilon > 0$, we can find $N \in \mathbb{Z}^+$ such that any $n > N, d(x_n, x) < \varepsilon$.

Lemma 2.1.2 (Limits are Unique)**.** *If a limit exists,* $\lim x_n = x = x'$*, then* $x' = x$.

Proof. Assume x, x' are both limits of $\{x_n\}$. Then, we can try to estimate $d(x, x') \leq d(x, x_n) + d(x_n, x')$.
Because $x_n \to x$, for any $\epsilon > 0$, we can find $N, n > N$, such that $d(x_n, x_0) < \epsilon$
For x', we can find the same $\epsilon > 0$, with N' such that and $n > N'$, we have $d(x_n, x_0) < \epsilon$.
Now, take $n > \max\{N, N'\}$, then we have $d(x_n, x_0) < \epsilon$ and $d(x_n, x') < \epsilon$.
Applying the triangle inequality again gives that $d(x, x') \leq d(x_n, x) + d(x_n, x') < 2\epsilon$. It follows from this that $d(x, x') < 0$. If it's in $\mathbb{R}$, so there's always some ϵ such that 2ϵ is smaller, which means they must be the same point, which in the metric space, is equivalent to saying $x = x'$. ☐

Every sequence that converges does so uniquely. But, sometimes sequences will diverge.

Example 2.2.3. $\{x_n = (-1)^n\}$ *is divergent.*

Proof. Assume $x_0 \in \mathbb{R}$ is a limit. We have that of $|x_0 - 1|, |x_0 + 1|$, at least one is not zero. Call this nonzero element ϵ_0. WLOG assume $\epsilon_0 = |x_0 - 1|$. Consider $\epsilon = \frac{1}{2}\epsilon_0$. For any N, we can find $n > N$. ☐

Proposition 2.3. $S \subseteq X$ *a point* $x \in X$ *is a limit point of* S *if any only if there exists some sequence* $\{x_n\}$ *in* S *with* $x_n \neq x$ *so that* $x_n \to x$.

Proof. $\Rightarrow$: Assume x is a limit point of S. Consider balls $B_{\frac{1}{n}}(x)$ contains some point $x_n \neq x$. Then, looking at the sequence $\{x_n\}, d(x_n, x) < \frac{1}{n}$. We can take $\forall \epsilon > 0$, N such that $\frac{1}{N} < \epsilon$, then any $n > N$, fulfills $\frac{1}{n} < \frac{1}{N} < \epsilon$. This proes that $x_n \to x$ as $n \to \infty$.
$\Leftarrow$. $x_n \to x$, $x_n \neq x$, $x_n \in S$. Take any $B_r(x)$, then $\exists N$ so that any $n > N$ implies $d(x_n, x) < r \Leftrightarrow x_n \in B_r(x)$, so we're done. ☐

A corollary of this is that if any convergent sequence has it's limit in S, then the set S must be closed.

Definition 2.12 (Sequential Compactness)**.** A subset $S \subseteq X$ is called **sequentially compact** if any sequence $\{x_n\}$ in S has a convergent subsequence in S, which is convergent in S.

Example 2.2.4. We've alreadyt shown that $\{x_n = (-1)^{n+1}\}$. But we can take a subsequence $x_{n_k} = (-1)^{2k+1} = -1\}$ converges to -1. Likewise for even integers it converges to one.

Example 2.2.5. Let $F \subseteq X$ $|F| < \infty$. Then, claim is that F is sequentially compact. Take some sequence $\{x_n\}$. We want to show it has a constant subsequence.

We are going to take a few lectures to prove the next result.

Theorem 2.2 (Heine-Borel)**.** $[0, 1] \subseteq \mathbb{R}$ *is sequentially compact.*

It's fairly intuitive to see why sets like $(0, 1)$ aren't sequentially compact, since this converges to 1, but 1 is not in the set.

Definition 2.13 (Compactness)**.** A subset $S \subseteq X$ is called compact if any open cover of S has a finite subcover.

Definition 2.14 (Open Cover)**.** An open cover $\{U_\alpha | \alpha \in \Lambda\}$ is a collection in X such that $\bigcup_{\alpha \in \Lambda} \supseteq S$.

Definition 2.15 (Finite Cover)**.** A cover with $|\Lambda| < \infty$.

Definition 2.16 (Subcover)**.** A subset $A \subseteq \Lambda$ of the index set, and take $\{U_\alpha | \alpha \in A\}$ is the union, but it also must cover S.

Theorem 2.3. (X, d)*.* $K \subseteq X$ *is compact if and only if* K *is sequentially compact.*

Proof. We will show the forward direction. Want to let $\{x_n\}$ a sequence in K. we want to show that there exists some subsequence $\{x_{n_k}\}$, so that x_{n_k} converges to $x_\infty \in K$ as $k \to \infty$. Assume for the sake of contradiction that no such subsequence exists. Then, any $x \in K$ is not a limit of a subsequence from $\{x_n\}$. But, we can find $r_x > 0$ so that $B_{r_x}(x) \cap K$ contains at most one point from $\{x_n\}$, which $\forall x \in K$, $\{B_{r_x}(x) | x \in K\}$ is an open cover of K.
K is compact, so it has a finite subcover, $x^1, x^2, \ldots, x^n$, so take $B_r x^n, \ldots,$ then

- Cover $K \subseteq \{x_n\}$.
- At most contains one point from $\{x_n\}$.

$\rightarrow \exists$ some $B_{r_{\tilde{x}}}(\tilde{x})$ $\tilde{x}$ from teh finite subcover. $\square$

Theorem 2.4. *$\{K_\alpha | \alpha \in \Lambda\}$ a collaction of compact subsets in (X, d). $\bigcap_{\alpha \in \Lambda} K_\alpha \neq \emptyset \Leftrightarrow$ for any finite subcollection $\Lambda_f \subseteq \Lambda$, $\bigcap_{\alpha \in \Lambda_f} \neq \emptyset$.*

3 Sequences And Series

3.1 Sequences

Proposition 3.1. *Let $x_n \to x$, $y_n \to y$. Then*

1. *$\{x_n \pm y_n\}$ is convergent and converges to $x \pm y$.*
2. *$\{x_n y_n\}$ is convergent and goes to xy.*
3. *If $y \neq 0$, then for large n, $\left\{\frac{x_n}{y_n}\right\}$ converges to $\frac{x}{y}$.*

1. Take $+$ as our example. Then $|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)|$. The argument we make is that because $x_n \to x$, $y_n \to y$, for any $\epsilon > 0$, we can find $N \in \mathbb{Z}^+$ such that $|x_n - x| < \frac{\epsilon}{2}$, and same for $y_n - y$. $\square$

2. The idea is we take $|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy| = |x_n(y_n - y) + (x_n - x)y| = |x_n||y_n - y| + |x_n - x||y|$, which gives the idea of the proof. Need to use the idea that x_n is bounded.

$\square$

The third proof is just more algebraic tricks.

Example 3.1.1. Prove that for $p > 0$, $\frac{1}{n^p} \to 0$.

Proof. If $p \in \mathbb{Z}^+$, then $0 \leq \frac{1}{n^p} \leq \frac{1}{n}$, which goes to 0 by the squeeze theorem.
If $p \in \mathbb{Q}^+$, then it's a fraction of 2 positive integers $\frac{\ell}{k}$ where $\ell, k \in \mathbb{Z}^+$. Then $\frac{1}{n^p} = \frac{1}{n^{\ell/k}} = \frac{1}{\sqrt[k]{n^\ell}}$. From the integer case, $1/n^\ell \to 0$, so the whole thing converges.
If $p \in \mathbb{R}^+$, we can define $n^p = \sup\{n^q | q \in \mathbb{Q}, q \leq p\}$. By this, we have $n^p \geq n^q$ for any rational $q \leq q$. Divide through, and apply the squeeze theorem, so we're done. $\square$

Recall, we had a few ways to determine whether sequence limits exist,

- Cauchy$\Rightarrow$ Limit
- $x_n \leq x_{n+1}$ with an upper bound gives limit exists.

If we want to be more general tho, we can construct the set $L = \{a \in \mathbb{R} \cup \{\pm\infty\} | x_{n_k} \to a\}$, the set of subsequence limits of $\{x_n\}$. E.g., if $\{x_n = -1^n\}$, then $L = \{1, -1\}$.

Definition 3.1. For the set L as defined above,

- $\limsup_{n\to\infty} x_n = \sup L$.
- $\liminf_{n\to\infty} x_n = \inf L$.

Note that if $\liminf x_n = \limsup x_n = a$, then $\lim x_n$ exists and is equal to a.

Lemma 3.0.1. *L is closed.*

Proof. To show L is closed, it's enough to show that any sequence $\{a_k\}$ from L that $a_k \to a$ implies $a \in L$. Consider the following.

$$\begin{aligned}
a_1 &\Rightarrow \exists x_{n_1} \text{ so that } |x_{n_1} - a_1| < 1 \\
a_2 &\Rightarrow \exists x_{n_2} \text{ so that } |x_{n_2} - a_2| < \frac{1}{2} \\
&\vdots \\
a_k &\Rightarrow \exists x_{n_k} \text{ so that } |x_{n_2} - a_k| < \frac{1}{k}
\end{aligned}$$

We now construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let's show that $x_{n_k} \to a$. We take $|x_{n_k} - a| \leq |x_{n_k} - a_k| + |a_k - a| = \frac{1}{k} + |a_k - a|$. The right hand side converges to 0 as $k \to \infty$, so $x_{n_k} \to a$. ☐

CAUTION, we cannot state that $\exists N.n > N$ gives $x_n \leq \sup L$. But it's ok for any $x > \sup L$. Consider $\{x_n + \frac{1}{n}\}$, we have $\limsup x_n = \lim x_n = 1$, but every element is larger than 1.

3.2 Series

We may want to think about series as a special case of sequences. Probably not

Definition 3.2. A series $\sum_{n=1}^{\infty}$ is convergent if the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ is convergent. Similarly for divergence.

Theorem 3.1 (Cauchy's Criterion)**.** *$\sum x_n$ is convergent if and only if for any $\epsilon > 0$, $\exists N$ such that $\forall n > N$, and any $p \geq 0$, $\left|\sum_{k=n}^{n+p} x_k\right| < \epsilon$.*

Proof. Consider the sequence $\{S_n\}$ of partial sums. Then, we take $|S_{n+1} - S_{n-1}| = \left|\sum_{k=n}^{n+p} x_k\right| < \epsilon$. This is equivalent to stating $\{S_n\}$ is cauchy in $\mathbb{R}$, which is equivalent to stating S_n is convergent (and thus the sum is convergent). In particular, take $p = 0$. Then $\sum_{k=n}^{n+p} x_k = x_n$. This leads us to a corollary. that if $\sum x_n$ convergent, then $x_n \to 0$. ☐

Proposition 3.2. *Let $\sum_{n=0}^{\infty} x_n$, with $x_n \geq 0$. This is convergent if and only if its partial sums is a bounded sequence.*

Proof. $\Leftarrow$. $x_n \geq 0 \Rightarrow S_n$ always increasing, which gives $S_n \leq S_{n+1}$. If $\{S_n\}$ bounded, $\{S_n\}$ convergent, so the series is convergent.
$\Rightarrow$. $\sum x_n$ convergent $\to \{S_n\}$ convergent, $\{S_n\}$ bounded. ☐

Definition 3.3. For any series $\sum_{n=0}^{\infty} x_n$, if $\sum_{n=0}^{\infty} |x|$ is convergent, then $\sum_{n=0}^{\infty} x_n$ absolutely convergent.

Proposition 3.3. *Absolute convergence implies convergence for any series.*

Proof.

$$\left|\sum_{n=N}^{N+p} x_n\right| \leq \sum_{n=N}^{N+p} |x_n| < \varepsilon$$

True by cauchy's criterion. ☐

Proposition 3.4 (Comparison Test)**.** *Consider $\sum x_n, \sum y_n$.*

1. *If we have $|x_n| \leq y_n$, and $\sum y_n$ is convergent, then $\sum x_n$ is absolutely convergent.*
2. *if $x_n \leq y_n$ and $\sum x_n$ diverges to $+\infty$, then $\sum y_n$ also diverges.*

Proof. 1.

$$\sum_{n=N}^{N+p} |x_n| \leq \sum_{n=N}^{N+p} = \left|\sum_{n=N}^{N+p} y_n\right| < \varepsilon$$

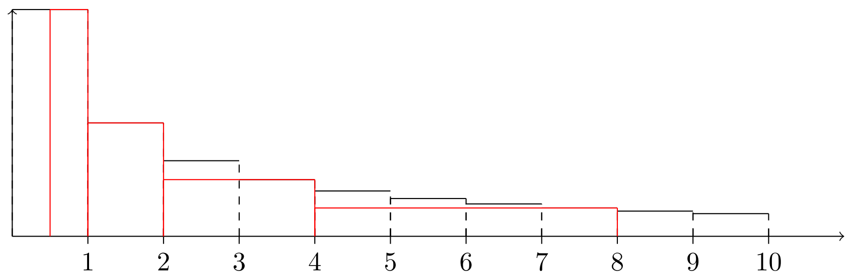
When $\sum y_n$ convergent.
2. We can just take the partial sums, and show relative divergence to $+\infty$. ☐

Example 3.2.1. Consider $\sum_{n=0}^{\infty} \frac{1}{n^2+2}$. We know that $\sum \frac{1}{n^2}$ is convergent, so by comparison, the first series is also convergent.
We could also consider $\sum \frac{n}{2^n}$ by comparing $n \ll 1.5^n$ for large n.

Proposition 3.5. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *convergent if* $p > 1$*, divergent if* $p \leq 1$.

Proof. If $p \leq 0$, $\frac{1}{n^p}$ not convergent, so neither is its sum. □

We can try comparing this to $\sum_{k=0}^{\infty} 2^k x_{2^k}$, which might look as



This will take

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \Rightarrow \sum 2^k \frac{1}{(2^k)^p} = \sum \frac{1}{2^{k(p-1)}} = \sum \left(\frac{1}{2^{p-1}}\right)^k$$

Also, we can compare this (though not as rigorous) to the integral of our series. Except we don't have a meaning for integration yet, so maybe hold off and we'll continue to use the series.

Theorem 3.2 (Root Test)**.** *Consider* $\sum x_n$*. We take* $\sqrt[n]{|x_n|} = \alpha$*. We have the following.*

1. $\alpha < 1$*, then* $\sum x_n$ *absolutely converges.*
2. $\alpha > 1$*, then* $\sum x_n$ *diverges.*
3. $\alpha = 1$*, then the test fails.*

Proof.
1. If $\alpha = \limsup_{n\to\infty} \sqrt[n]{|x_n|} < 1$. We should try to find some known series which converges to β, which gives $\alpha < \beta < 1$. This inequality gives us $\exists N$ such that $n > N$, we have $\sqrt[n]{|x_n|} < \beta$, or $|x_n| < \beta^n$, but $\beta < 1$, so it's absolutely convergent.
2. We can just choose a subsequence such that $|x_{nk}| \geq 1$, so it must diverge. No bueno.
3. Fails, which you can show by just taking $\frac{1}{n^p}$ as our sequence.

□

Theorem 3.3 (Ratio Test)**.** $\sum x_n$. $\limsup_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \alpha$.

1. $\alpha < 1$*,* $\sum x_n$ *absolutely converges.*
2. $\alpha > 1$*,* $\sum x_n$ *diverges.*
3. $\alpha = 1$*, the test fails.*

Proof.
1. If $\alpha < 1$, find β so that $\alpha < \beta < 1$, then $\exists N, n > N$ $\left|\frac{x_{n+1}}{x_n}\right| < \beta$ which gives $|x_{n+1}| < \beta|x_n|$so $|x_{N+1} < \beta|x_N|$, etc etc etc. Already proved this in the homework I'm pretty sure. Basically, we have $|x_{n+p}| < \beta^p|x_N|$ which implies convergence, since $\beta < 1$.
2. By a similar argument, we have divergence. Take some subsequence such that $1 < \beta < \alpha$.
3. Failure. Ta

□

Theorem 3.4. *If* $\sum a_n$ *converges absolutely to* A*, and* $\sum b_n$ *converges to* B*, then* $\sum_{k=0}^{n} a_k b_{n-k}$ *converges to* AB*.*

Proof.

$$S_m = \sum_{n=0}^{m} C_n = \sum_{n=0}^{m}\sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{m} a_k \sum_{n=0}^{m-k} b_n$$

Now, we write

$$\begin{aligned} S_m - AB &= \sum_{k=0}^{m} a_k S^b_{m-k} - AB \\ &= \sum_{k=0}^{m} a_k (S^b_{m-k} - B) + B\left(\sum_{k=0}^{m} a_k - A\right) \end{aligned}$$

Now, it's sufficient to show that

$$\sum_{k=0}^{m} |a_k||S^b_{m-k} - B| \to 0$$

We introduce the notation $\beta_n = S^b_n - B$. Then, we have

$$\sum_{k=0}^{m} |a_k||\beta_{m-k}| = |a_0||\beta_m| + |a_1||\beta_{m-1}| + |a_2||\beta_{m-2}| + \cdots + |a_m||\beta_0|$$

We know that for any $\varepsilon > 0$, we can find N such that $n > N$, $|\beta_n| < \varepsilon$. Choose some such N. We can split our partial sums into the following

$$\begin{aligned} = \big(|a_0||\beta_m| + a_1||\beta_{m-1}| + \cdots + |a_{m-(N+1)}||\beta_{N+1}|\big) \\ + (|a_{m-N}||\beta_N| + \cdots + |a_m||\beta_0|) \end{aligned}$$

Every β_k where $k < N$ will be smaller than ε, so we can rewrite this as

$$\begin{aligned} &= \big(|a_0||\beta_m| + a_1||\beta_{m-1}| + \cdots + |a_{m-(N+1)}||\beta_{N+1}|\big) \\ &\quad + (|a_{m-N}||\beta_N| + \cdots + |a_m||\beta_0|) \\ &\le \varepsilon \cdot \sum_{k=0}^{m-(N+1)} |a_k| + \max\{|\beta_0|, |\beta_1|, \ldots, |\beta_N|\} \sum_{k=m-N}^{m} |a_k| \end{aligned}$$

But $\sum a_k$ is absolutely convergent, so we have that ε on the left multiplied by a constant is still bounded. We also know that the term on the right must converge to zero, by Cauchy's Criterion, which means we can essentially replace it by an appropriate choice of ε. So, now we have that

$$\sum_{k=0}^{m} |a_k||\beta_{m-k}| \le \varepsilon \left(\sum |a_k| + \max\{|\beta_0|, \ldots, |\beta_m|\}\right)$$

which shows that for $m \to \infty$, the sum converges to 0. So we're done! □

Theorem 3.5 (By Abel)**.** *If $\sum a_n = A$, $\sum b_n = B$, and $\sum_{k=0}^{n} a_k b_{n-k} = C$, then $C = AB$.*

We will prove this later using power series. Stay tuned kiddos!
Also, rearrangement is weird. We'll make that statement rigorous.

Definition 3.4. A rearrangement of $\sum_{n=1}^{\infty} a_n$ is a series $\sum_{n=1}^{\infty} a_{r(n)}$ with $r : \mathbb{Z}^+ \to \mathbb{Z}^+$ where r is bijective.

Some series may have different limits under different arrangements. Consider

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \left(0 - 1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{3} - \frac{1}{5}\right)$$

which gives a negative number, as opposed to our usual result 2.

Theorem 3.6 (By Riemann)**.** *If $\sum x_n$ converges, but not absolutely, then for any $a \le b(\pm\infty)$. There is some rearrangement with S'_n partial sums so that $\liminf_{n\to\infty} S'_n = a; \limsup_{n\to\infty} S'_n = b$.*[6]

In particular, if $a = b$, then there's a convergent rearrangement. There's a proof of this in rudin. We're going to skip the proof. I'm going to try to write it up.

Theorem 3.7. *If $\sum x_n$ converges absolutely, then any rearrangement will converge to the same number.*

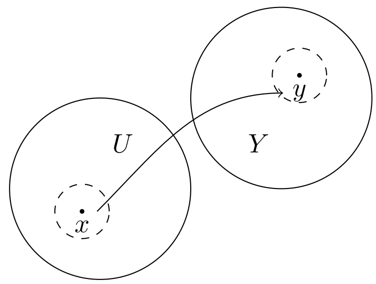
[6] oh BOY this is trippy

4 Limits and Continuity

4.1 Limits

Consider $(X, d_X), (Y, d_y)$ metric spaces, with some $(U, d_X|_U) \subseteq (X, d_X)$. A function, or a map is some $f : U \to Y$.

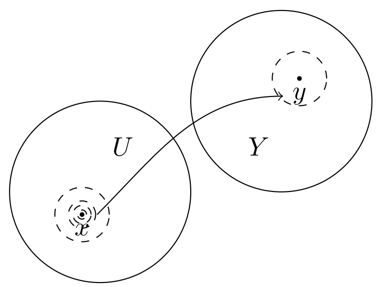
Definition 4.1 (Limit at a Limit Point)**.** Consider $(X, d_X), (Y, d_Y)$ metric spaces, with $U \subseteq X$. Then, for some $f : U \to Y$, $\lim_{x \to x_0} f(x) = y_0 \in Y$ if for any $\varepsilon > 0$, there exists some $\delta > 0$ such that any $x \in U$ with $0 < d_X(x, x_0) < \delta$, has $d_Y(f(x), y_0) < \varepsilon$.



Points in the ball map to points in the other ball.

Example 4.1.1. Check that $\lim_{x \to 0} x = 0$.
For any $\varepsilon > 0$, we want $\delta > 0$ so that whenever $0 < |x - 0| < \delta$, we have $|f(x) - 0| < \varepsilon$. Take $\delta = \varepsilon$, and we're done.

Proposition 4.1. *1.* $\lim_{x \to x_0} f(x) = y_0 \Leftrightarrow$ *any sequence* $\{x_n\}$ *in* $U \setminus \{x_0\}$ $x_n \to x_0$*, we will have* $f(x_n) \to y_0$*.*



You just ε-δ your way through this for the forward direction. For the backwards direction, we assume a contradiction.

Proposition 4.2. $\lim_{x \to x_0} f(x)$ *is unique if it exists.*

Proof (Method 1). Assume that $y_1, y_2 \in Y$ such that $y_1 = y_2 = \lim_{x \to x_0} f(x) = y_1 = y_2$. Then, we have that $\forall \varepsilon > 0, \exists \delta > 0$, so that any $0 < d_X(x, x_0) < \delta_1$, $d_Y(f(x), y_1) < \varepsilon$. We can say something similar about δ', y_2, which means that $d_Y(y_1, y_2) \leq d_Y(y_1, f(x)) + d_Y(y_2, f(x)) = 2\varepsilon$. So $d_Y(y_1, y_2)$ must be zero. □

Proof (Method 2). Assume that $y_1, y_2 = \lim_{x \to x_0} f(x)$ Subtract the two from each other and we're one by the uniqueness of sequential limits. □

We can also define all the usual stuff about functions. When $f : X \to \mathbb{R}$, we have

- $(f \pm g)(x) = f(x) \pm g(x)$.
- $(f \cdot g)(x) = f(x) \cdot g(x)$.
- $(f \div g)(x) = f(x) \div g(x)$ when $g(x) \neq 0$.

We also have the following, for $\lim_{x \to x_0} f(x) = F, \lim_{x \to x_0} g(X) = G$,

1. $\lim_{x \to x_0} (f \pm g)(x) = F \pm G$.
2. $\lim_{x \to x_0} (f \cdot g)(x) = F \cdot G$.
3. $\lim_{x \to x_0} (f \div g)(x) = F \div G$ when $G \neq 0$.

We can prove these things using sequential limits, by taking $x_n \to x_0, x_n \neq x_0$, and taking $(f+g)(x_n) = f(x_n) + g(x_n)$.The advantage here is that it lets us skip $\delta - \varepsilon$ style arguments.

Example 4.1.2 (Polynomial Gang)**.** All $f : \mathbb{R} \to \mathbb{R}$.

1. $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} x = x_0$.
2. $f(x) = 3x^2 + 2x - 3$. We need only evaluate $\lim_{x\to x_0} 3x^2 = 3\lim_{x\to x_0} x^2 = 3\lim_{x\to x_0} x^2 \lim_{x\to x_0} x^2 = 3x_0^2$.[7]
3. Consider $\frac{f(x)}{g(x)}$ some rational function where f, g are two polynomials. Let $U \subseteq \mathbb{R}$ where $U = \mathbb{R} \setminus \{x | g(x) = 0\}$. Then, $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{f(x_0)}{g(x_0)}$ when $x_0 \in \mathbb{U}$.

4.2 Continuity in Metric Spaces

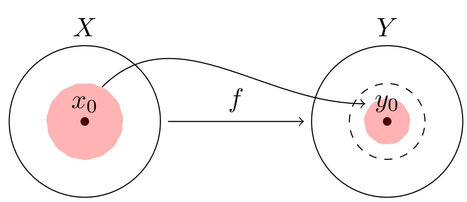
Consider $(X, d_x), (Y, d_y)$ metric spaces.

Definition 4.2 (Continuity)**.** Then

1. If x_0 is an isolated point in X, we define $f(x)$ continuous at x_0.
2. If x_0 is not isolated, then we say $f(x)$ is continuous at x_0 if any sequence $x_n \to x_0$, there is $f(x_n \to f(x_0)$ as $n \to \infty$

Proposition 4.3. *$f : X \to Y$, $x_0 \in X$. f is continuous at $x_0 \Leftrightarrow$ for any $\varepsilon > 0, \exists \delta > 0$ such that any $d_x(x, x_0) < \delta$ there is $d_y(f(x), f(x_0)) < \varepsilon$.*

Proof. $\Rightarrow$. If x_0 isolated, for any $\varepsilon > 0$, we can find $\delta > 0$, only x_0 satisfies it, we're good, since $x = x_0 \Rightarrow f(x) = f(x_0)$.[8] $\square$



Proposition 4.4. *If f, g are continuous at $x_0 \in X$, we have*

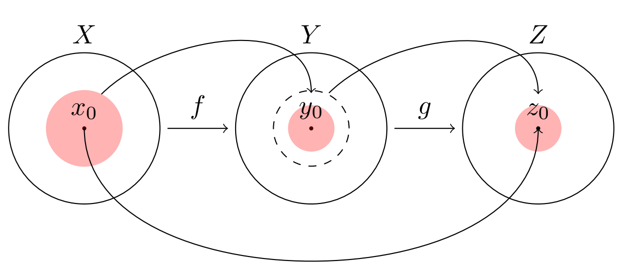
1. *$f \pm g, f \cdot g$ are continuous at x_0.*
2. *when $g(x_0) \neq 0$, $f \div g$ is continuous at x_0.*

The proof for this follows the proof for limits and is pretty easy.

Example 4.2.1. We have some examples from what we've shown.

1. Polynomials, rational functions, $e^x, \cos(x), \sin(x)$ are continuous at anhy point in the natural domain.
2. $+, -, \cdot, \div$ of these functions are continuous everywhere in their natural domains.

Proposition 4.5. *Consider $(X, d_x), (Y, d_y), (Z, d_Z)$. Consider $f : X \to Y, g : Y \to Z$. Let $f(x_0) \in Y$ continuous, $g(f(x_0))$ is continuous. This function is called $g \circ f : X \to Z$, and it is continuous at x_0.*



[7]The general statment is that $\lim_{x\to x_0} f(x) = f(x_0)$, where f is a polynomial is equivalent to saying that any polynomial is continuous on $\mathbb{R}$.

[8]there's more I missed here, check her notes

We can also take more general f, for instance, $d_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}$ is also continuous.
Or, consider $g_1(x) = 2x$,g $_2(x) = 3x$ from $\mathbb{R} \to \mathbb{R}$. Then, define $g : \mathbb{R} \to \mathbb{R}^2$ so that $x \mapsto (g_1(x), g_2(x))$.
Now, take $f \circ g : \mathbb{R} \to \mathbb{R}$, we have $f \circ g(x) = f(g(x)) = f(2x, 3x) = 2x + 3x = 5x$.

Definition 4.3. $f : X \to Y$ is called a continuous function (or a continuous map) if f is continuous at any $x \in X$.

Theorem 4.1. $f : X \to Y$

1. *f is continuous if and only if the preimage of any open subset of Y is open in X.*

2. *f is continuous if and only if the preimage of any closed subset of Y is closed in X.*

We will only prove the first case. The second case is exactly the same.

Proof (1). $\Rightarrow$. Take any open set $V \subseteq Y$, with $f^{-1}(V) = \{x \in X | f(x) \in V\}$. We will show that $f^{-1}(V)$ is open, i.e. any $x \in f^{-1}(V)$ is an interior point.
Take some $f(x) \in V$ which is an interior point in V. Then, there exists some $\varepsilon > 0$ so that $B_\varepsilon(f(x)) \subseteq V$. Then, by the continuity of f at x, we can find $\delta > 0$ so that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. This implies that $B_\delta(x) \subseteq f^{-1}(V)$, which shows x is an interior point of the preimage.
$\Leftarrow$. This is trivial. We've already shown any preimage of an open set is open. Then, we just do something very very similar. Take any point $x \in X$, and map it to $f(x)$, where $f(x) \in Y$. Consider $B_\varepsilon(f(x))$, then the preimage in X is open by our assumption of continuity. In particular, $x \in f^{-1}(B_\varepsilon(f(x)))$ i san interior point. ☐

Definition 4.4. $x_0 \in X$ is discontinuous point of f if f is not continuous at x_0.

If we restrict ourselves to $\mathbb{R}$ and intervals thereof, we have that there are different limits from the left or the right. These are defined by

$$\lim_{x \to x_0^-} f(x) : \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < x_0 - x < \delta$$
$$\lim_{x \to x_0^+} f(x) : \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < x - x_0 < \delta$$

Definition 4.5. Assume x_0 is a discontinuous point for f on some interval in $\mathbb{R}$.

1. Call x_0 the first kind of discontinuous point if both $\lim_{x \to x_0^\pm} f(x)$ exist, and $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+}$. such x_0 is called a **removable discontinuous point**.

2. Call x_0 the second kind of discontinuous point if it is not the first kind.

If x_0 is removable, then we can modify f by defining $f(x_0) = \lim_{x \to x_0} f(x)$, and then f is continuous.
An interesting weird case to consider is

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

4.3 Continuity and Compactness

Let $X \xrightarrow{f} Y$ continuous.

Theorem 4.2. *Assume f continuous. Then for any compact subset $K \subseteq X$, the image of K, $f(K) = \{f(x) | x \in K\}$ is compact in Y.*

Proof (Compactness). Take any open cover $\{V_\alpha | \alpha \in \Lambda\}$ of $f(K)$. Let's denote $f^{-1}(V_\alpha) = U_\alpha \subseteq X$ is open because f is continuous, and $\{U_\alpha | \alpha \in \Lambda\}$ is a cover of K. The compactness of K implies there exists a finite subcover from $\{U_\alpha | \alpha \in \Lambda\}$. By this then, $\{V_\alpha | \alpha \in \Lambda'\}$ where Λ' is finite is a finite subcover. This follows pretty directly from theorem 4.1. ☐

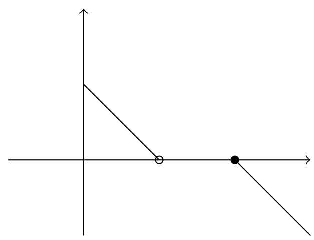
Proof (Sequential Compactness). Assume K is sequentially compact. Take any sequence $\{x_n\}$. Seqeuential compactness implies $\exists \{x_{n_k}\}$ su that $x_{n_k} \to x_0 \in K$. Continuity of f implies that $f(x_{n_k}) \to f(x_0) \in f(K)$. By this way we get such a subsequence $f(x_{n_k})$ of $f(x_n)$ which is sequentially compact in $f(K)$. ☐

Definition 4.6. A function f is called **proper** if the preimage of any compact set is compact

Not every continuous map is proper. For example, $f(x) = 0$, $x \in \mathbb{R}$. $f^{-1}(0) = \mathbb{R}$ is not compact. A less trivial example is $f(x) = \frac{1}{x}, x \in (0, +\infty)$. We have $[0, 1] \subseteq \mathbb{R}$ is compact, but $f([0, 1]) = [1, +\infty)$ is not compact. Spiced!

Corollary 4.2.1. *Any real valued function defined on a compact subset of X can obtain it's supremum and infimum (e.g. has a maximum and a minimum).*

A logical question we might want to know is whether f^{-1} is continuous for any continuous map. If we want to say the answer is no, it has to be injective but not open. Consider this function



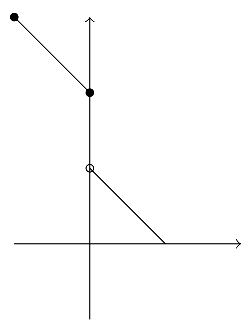
The function

$$f(x) = \begin{cases} 1 - x & 0 \leq x < 1 \\ 2 - x & 2 \leq x \leq 3 \end{cases}$$

where $f : [0, 1) \cup [2, 3] \to \mathbb{R}$ injects. $\left(-\frac{1}{2}, \frac{1}{2}\right) \subseteq \mathbb{R}$ is open. Then, we have $f^{-1}\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right) = \left(\frac{1}{2}, 1\right) \cup \left[2, \frac{5}{2}\right)$ which is open in the domain.
Now, if we consider the inverse function,

$$f^{-1}(x) \begin{cases} 2 - x & -1 \leq x \leq 0 \\ 1 - x & 0 < x \leq 1 \end{cases}$$

we have



which is discontinuous at $x = 0$.

Definition 4.7. If we assume X, Y two metric spaces, with $f : X \to Y$ bijective, $f^{-1} : Y \to X$ defined, and both f^{-1}, f are continuous, thhen these two spaces are then called **homeomorphic** to each other.

Note, if X, Y are both homeomorphic, then they have the same topology.

Theorem 4.3. *Assume $f : X \to Y$ bijective and continuous. Then, if X is compact, the inverse map $f^{-1} : Y \to X$ is also continuous. (I.e. X, Y are homeomorphic).*

Proof (Compactness). We can take f^{-1} continuous$\Leftrightarrow$ f is open. Take any $U \subseteq X$ open. Then, it's sufficient to show that $f(U) \subseteq Y$ is also open. We will show that $Y \setminus f(U)$ is closed. We have $f(X \setminus U) = Y \setminus f(U)$, by the bijectivity of f. By the continuity of f, we have that $f(X \setminus U)$ is compact, and thus closed in Y, which completes the proof. ☐

Proof (Sequential Compactness). Want to show that f^{-1} is continuous. Take any sequence $\{f(x_n) | x_n \in X\}$ in Y, with $f(x_n) \to y_0 \in Y$. Since our map surjects, we have $f(x_0) = y_0$. A really reasonable idea should be that $f^{-1}(f(x_n)) = x_n$, and $f^{-1}(f(x_0)) = x_0$. Now, we want to demonstrate that $x_n \to x_0$. Since X is sequentially compact, we have $\{x_{n_k}\}$ converges to x' in X. By the continuity of f, we have $f(x_{n_k}) \to f(x')$. But since $f(x_0)$ has only one limit, we have $x' = x_0$, so every subsequential lmiit of x_n is the same, and thus $x_n \to x_0$. ☐

Furthermore, if $x_n \not\to x_0$, then $\exists \varepsilon > 0$ so that $\forall N, n > N$ $d(x_n, x_0) \geq \varepsilon_0$. Then, we have a spooky subseqeuence x_{n_k} with the bad limiting property, since it's not cauchy or something. Spicy but the lecture is over.

4.3.1 Uniform Continuity

Typically, we have with our ϵ-δ definition, δ has x_0, ϵ dependence. Sometimes this isn't the case though, which is nice.

Definition 4.8. $f : X \to Y$ is called **uniformly continuous** if for any $\epsilon > 0$, $\exists$ some $\delta > 0$ so that any two points $x_1, x_2 \in X$ with $d_x(x_1, x_2) < \delta$, then $d_y(f(x_1), f(x_2)) < \epsilon$ (i.e. has no x_0 dependence).

It's also going to be a homework problem to show that $d_X : X \times X \to \mathbb{R}$ is continuous.

4.4 Continuity and Connectedness

Definition 4.9. Let X be a metric space. X is called **connected** if any subset $S \subseteq X$ which is both open and closed is either $S = \emptyset, S = X$.[9]

Example 4.4.1. Consider the ball $B_r(x_0) \subseteq \mathbb{R}^n$. This is connected.

Example 4.4.2. Consider 2 disjoint balls $B_r(x_0), B_s(x_1)$ with $B_r(x_0) \cap B_s(x_1) = \emptyset$, their union is not connected.

Proposition 4.6. *X is connected if and only if $X = U \cup V$ with $U \cap V = \emptyset$, both U, V are open implies that one of U, V is empty.*

Proof. It follows directly from the definition, more or less. Consider the subset U. Then $U = X \setminus V$. Since U is open, V is closed. Since we assumed X is connected, one of these has to be empty. $\square$

Theorem 4.4. *Consider the continuous map $f : X \to Y$, where X is connected, then $f(X) \subseteq Y$ is also connected (as a metric space with the induced metric from Y).*

Proof. We're just going to decompose $f(X) = U \cup V$ with both U, V open, and $U = \tilde{U} \cap f(X)$ the intersection of open set $\tilde{U}$ with $f(X)$, and likewise for V. Then, we take $f^{-1}(\tilde{U}) = f^{-1}(U)$ is an open set in X, and likewise for V by the continuity of X. Then, if we take the union of the two, we have the entire space X. Since X is connected, we have $f^{-1}(\tilde{U}) \cap f^{-1}(\tilde{V}) = \emptyset$. This basically just yeets us the result. One of the preimages is empty, so one of the images is also empty, whcih is the desired result. $\square$

Example 4.4.3 (Topologists sin function)**.** consider $S = \{(x, \sin(\frac{1}{x})|x \neq 0\} \cup \{0\} \times [-1, 1] \subseteq \mathbb{R}^2$. We can prove that this is indeed connected on this interval (but not path connected).

Definition 4.10. A metric space X is called **path-connected** if for any two points, $x_0, x \in X$, there is a continuous map $f[0, 1] \to X$ so that $f(0) = x_0, f(1) = x_1$.

Theorem 4.5. $[0, 1] \in \mathbb{R}$ *is both connected and path connected.*

Proof.
1. Path Connected. Consider $f(t) = (1 - t)x_0 + tx_1$. We are done.
2. Connected. Consider $[0, 1] = A \cup B$, and assume A, B are both open and disjoint. Take $a \in A, b \in B$, with $a < b$. Then, consider $x_0 = \sup([a, b] \cap A)$. We know since A is open, $x_0 \notin A$. Similarly, it doesn't live in B. So, either A or B is open. So $[0, 1]$ is connected.

$\square$

Theorem 4.6. *For any metric space X, if X is path connected, X is connected.*

Proof. Consider $X = U \cup V$, with U, V open and disjoint. If neither is empty, take $x_0 \in U, x_1 \in V$. The assumption that X is path connected allows us to find a continuous map $f : [0, 1] \to X$ so that $f(0) = x_0, f(1) = x_1$. Consider $f^{-1}(U), f^{-1}(V)$ are both open in $[0, 1]$, and their union is $[0, 1]$. One of these preimages is empty. Thus, one of U, V must be empty. If we consider $f^{-1}(U) = \emptyset$, then we have $f([0, 1]) \in V$, which is a contradiction to the assumption $f(0) = x_0 \in U$. $\square$

Theorem 4.7. *In $\mathbb{R}$, a subset is connected iff it is also path connected.*

Theorem 4.8 (Intermediate Value Theorem)**.** *Consider $[a, b] \subseteq \mathbb{R}$, with $f : [a, b] \to \mathbb{R}$ a continuous map. If $f(a) < f(b)$, then for each $f(a) < y_0 < f(b)$, there is some $x_0 \in (a, b)$ such that $f(x_0) = y_0$. It's the freakin vertical line test*

Proof. Consider $[a, b]$ connected. So, $f([a, b])$ is a conneceted subset of $\mathbb{R}$, and $f(a), f(b) \in f([a, b])$. So, we just do precisely the same thing we did earlier. $\square$

We also started monotonicity.

[9] would a better phrasing of this be that there are no clopen, nonempty proper subsets of X? Yes. She literally just did it on the board.

5 Differentiation

There's a conception of smoothness. Picture here.

Definition 5.1. $f : [a, b] \to \mathbb{R}. x_0 \in [a, b]$. Such f is differentiable at x_0 if the limit of the function $\phi_{x_0}(x) = \frac{f(x)-f(x_0)}{x-x_0}, x \in [a, b], x \neq x_0$ exists. We denote this limit as $f'(x_0)$, or $\frac{df}{dx}(x_0)$.

Example 5.0.1. Consider $f(x) = x^2$, then $\phi_{x_0}(x) = \frac{x^2-x_0^2}{x-x_0} = \frac{(x+x_0)(x-x_0)}{x-x_0} = x + x_0$, so in the limit $x \to x_0$, we get $2x_0$.

Example 5.0.2. Consider $f(x) = |x|$. We want to check our $x = 0$ is not differentiable. So $\phi_0(x) = \frac{f(x)-f_0}{x-0} = \frac{|x|}{x}$ These just dont approach the same thing from the left and the right, which is very spiced.

Proposition 5.1. *$f : [a, b] \to \mathbb{R}$, $x_0 \in [a, b]$. If f is differentiable at x_0, then f must be continuous at this point.*

Proof. We can jusst take

$$\lim_{x\to x_0} |f(x) - f(x_0)| = \lim_{x\to x_0} \left(\left| \frac{f(x) - f(x_0)}{x - x_0} \right| |x - x_0| \right) = f'(x_0) \lim_{x\to x_0} |x - x_0| = 0$$

so $\lim_{x\to x_0} f(x) = f(x_0)$, so f is continuous at x_0. $\square$

Now, we introduce some notation. If a function $f : X \to \mathbb{R}$ is continuous on X, we write $f \in C^0(X)$. If $f : X \to \mathbb{R}$ is differentiable at $x_0 \in X$, then $f'(x)$ is a function defined on X. If the derivative is a continuous function, we call $f' \in C^1(X)$.

Theorem 5.1. *$f, g : [a, b] \to \mathbb{R}$ both differentiable at $x_0 \in [a, b]$, then $f \pm g$, $f \cdot g$, f/g are all differentiable at x_0.*[10] *Also*

1. $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$.
2. $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.
3. $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}$.

As an example, we will prove 2.

Proof. We have

$$\begin{aligned}\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} &= \frac{(f(x)g(x) - f(x_0)g(x)) + (f(x_0)g(x) - f(x_0)g(x_0))}{x - x_0} \\ &= \frac{f(x) - f(x_0)}{x - x_0} g(x) + \frac{g(x) - g(x_0)}{x - x_0} f(x_0)\end{aligned}$$

We know that g is continuous at x_0, so we take $x \to x_0$, and we're left with

$$= f'(x_0)g(x_0) + g'(x_0)f(x_0)$$

If the target space is not commutative, we can still just be more careful about our multiplication order to write

$$= f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

$\square$

For 3, we need to consider the neighborhood $B_\varepsilon(x_0)$ upon which no $g(x) = 0$ for $x \in B_\varepsilon(x_0)$. Rui did it in lecture but I really don't feel like I need to. You can derive it from 2 after finding $\frac{d}{dx}\left(\frac{1}{g}\right)$.
We get a *lot* of mileage out of these definitions.

Example 5.0.3. Consider $f(x) = \sum_n a_n x^n$ where we assume that the relevant $a_n \neq 0$. Then, $f'(x) = (a_n x^n)' + \left(a_{n-1}x^{n-1}\right) + \dots$. Now, for each $(a_k x^k)' \overset{(2)}{=} a_k (x^k)'$. We can use induction here. $k = 0, 1$ is very easy. We can actually use the product rule here, which makes it easier to do induction. e.g. $k = 2 \Rightarrow (x^2)' = (x \cdot x)' = x' \cdot x + x \cdot x' = 2x$. For $k = 3 \Rightarrow (x^3) = (x^2)'x + x^2 x' = 3x^2$. We can yeet this into an inductive proof. We have $(x^k)' = (x^{k-1}x)' = (x^{k-1})'x + x^{k-1}x' = (k-1)x^{k-2}x + x^{k-1} = kx^{k-1}$, so we're done.

[10] making the assumption $g(x_0) \neq 0$ for the case of division.

We can generalize these polynomials to rational functions though, with f, g polynomials.

Theorem 5.2 (Chain Rule)**.** *Consider $f : [a,b] \to \mathbb{R}$ differentiable at $x_0 \in [a,b]$. Considedr $g : [c,b] \to \mathbb{R}$ with $f(x_0) \in [c,d]$, and g differentiable at $f(x_0)$. Then, $g \circ f : [a,b] \to \mathbb{R}$ is also differentiable at x_0. Moreover, $(g \circ f)' = g'(f(x_0))f'(x_0)$.*

Proof. First, we introduce some notation. If a function $p(x)$ has $\lim_{x \to x_0} p(x) = 0$, then $p(x) = o(|x - x_0|)$.
We simply take

$$\frac{g \circ f(x) - g \circ f(x_0)}{x - x_0}$$

First, we note that $g \circ f(x) - g \circ f(x_0) = g(f(x)) - g(f(x_0))$. Then, we know that $\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)}$ exists. So, we can write, where the little o functions come from not taking a limit, but making an estimate of the difference which we can limit later.

$$\begin{aligned} g \circ f(x) - g \circ f(x_0) &= g(f(x)) - g(f(x_0)) \\ &= (g'(f(x_0)) + o(|f(x) - f(x_0)|))\,(f(x) - f(x_0)) \\ &= (g'(f(x_0)) + o(|f(x) - f(x_0)|))(f'(x_0) + o(|x - x_0|))(x - x_0) \end{aligned}$$

In order to calculate the derivative then, we just want

$$\lim_{x \to x_0} ((g'(f(x_0)) + o(|f(x) - f(x_0)|))(f'(x_0) + o(|x - x_0|)))$$

Since $f(x)$ is continuous at x_0, we have $o(|f(x) - f(x_0)|) = 0$ in the limit, so we have

$$\lim_{x \to x_0} (g'(f(x_0))f'(x_0))$$

So we're done. ∎

So, we can use the chain rule to do stuff, which is pretty exciting.

Example 5.0.4. If we have

$$f(x) = \begin{cases} x^2 & x < 0 \\ 0 & x \geq 0 \end{cases}$$

Then the derivative actually exists, because it's differentiable. The derivative is also continuous, but there's a sharp point at 0. So it's not twice differentiable.

Example 5.0.5. This function is apparently differentiable everywhere:

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$f(x) = x^2 \sin(\frac{1}{x})$ in a neighborhood of x. So, we have to explicitly calculate the derivative at $x = 0$ by

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \frac{x^2 \sin(\frac{1}{x})}{x} = x \sin(\frac{1}{x}) = 0$$

So, we have

$$f'(x) \begin{cases} 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This function is discontinuous at $x = 0$, with a second kind discontinuity. The derivative exists, but is discontinuous.

Example 5.0.6. We have checked that

$$f(x) = \begin{cases} x \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous on $\mathbb{R}$. It is not differentiable at 0, which is easily checked by taking the limit definition at $x = 0$, and noting that $\sin(\frac{1}{x})$ diverges.

5.1 Mean Value Theorem

Definition 5.2. $f : X \to \mathbb{R}$, where X is some metric space, with $x_0 \in X$. f has a **local maximum** at x_0 if $\exists B_\delta(x_0)$ such that $\forall x \in B_\delta(x_0)$, $f(x_0) \geq f(x)$. Similarly for local minimum, with the opposite inequality.

Definition 5.3. $f : [a, b] \to \mathbb{R}$. A point $x_0 \in [a, b]$ is called a **critical point** if f is not differentiable at this point, or $f'(x_0) = 0$.

Theorem 5.3. *Assume f is defined on $[a, b]$. If f has a local maximum or minimum, at some point $x_0 \in (a, b)$, then x_0 must be a critical point.*

Proof. If f is not differentiable at x_0, then by definition x_0 is a critical point. Assume f is differentiable then. WLOG, we assume x_0 is a local maximum. It's just definition bashing, I got distracted by a really cool quantum problemset. □

Theorem 5.4 (Rolle's Theorem)**.** *$f(x)$ is continuous over $[a, b]$ and differentiable over (a, b), with $f(a) = f(b)$, then we can conclude there must be some point $x_0 \in (a, b)$ such that $f'(x_0) = 0$.*

To prove the intermediate value theorem, I wonder if it's ok to just linearly transform your straight line between A, B to get the new function which we can apply Rolle's theorem to. Rui wants us to take

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}x$$

I think this is equivalent to what I was saying.

Theorem 5.5 (Cauchy's Mean Value Theorem)**.** *For $(f(x), g(x))$ continuous on $[a, b]$ and differentiable on $[a, b]$, there exists some $x_0 \in (a, b)$ so that $(f(b) - f(a))g'(x_0) = (g(b) - g(a))f'(x_0)$.*

Some point on this curve is going to have the slope between these two lines. We can also imply this by Rolle's theorem, by defining

$$h(x) = (f(b) - f(a))g(x) - (g(b) - b(a))f(x$$

Theorem 5.6. *If f is differentiable on (a, b), then*

1. *$f'(x) \geq 0 \Rightarrow f$ increasing*
2. *$f'(x) \leq 0 \Rightarrow f$ decreasing*
3. *$f'(x) = 0 \Rightarrow f$ constant*

Theorem 5.7. *Consider f differentiable over $[a, b]$, with $f'(a) < f'(b)$. Then, for any $f'(a) < \mu < f'(b)$, there must exist some $x_0 \in (a, b)$ such that $f'(x_0) = \mu$.*

Proof. Consider the function $g(x) = f(x) - \mu x$. This is differentiable over $[a, b]$. $g'(x) = f'(x) - \mu$. Then $g'(a) = f'(a) - \mu < 0$ and $g'(b) = f'(b) - \mu > 0$, so there's some point at which $g'(x) = 0$, by the continuity of g'. It also lives in hte interior of the interval, by the argument presented in lecture. □

Corollary 5.7.1. *If f' is not continuous at some $x_0 \in (a, b)$, the discontinuity must be of the second kind.*

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